A Note of Introduction to Smooth Manifolds

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This is a note of John. Lee's Introduction to Smooth Manifolds, GTM 218. Differential Manifolds is a course I did not take at university. For taking the course Lie Group and Lie Algebra, I audited Differential Manifolds, while I found that Introduction to Smooth Manifolds is more readable than taking a course. Thus, I read this book. In this note, I mainly record the basic concepts, main theorems, and my thoughts and solutions for the problems and exercises in this book.

Pre-requisites of this note include a knowledge of the basic concepts of general topology, multivariate calculus, ordinary differential equation, partial differential equation.

Contents

1 Smooth Manifolds

1.1 Concepts

Definition (Topological Manifold (with Boundary)). *M is topological manifold of dimension n if it has the following properties:*

1. *M is a Hausdorff space;*
2. *M is second-countable;*
3. *M is locally homeomorphic to an open subset of $\mathbb{R}^n$ (or $\mathbb{H}^n$).*

Definition (Coordinate Chart). *Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} \in \mathbb{R}^n$.*

Definition (Smooth Atlas). *A collection of coordinate charts whose domains cover topological n-manifold M is called an atlas. An atlas $\mathscr{A}$ is called a smooth atlas if any two charts in $\mathscr{A}$ are smoothly compatible with each other.*

Definition (Smooth Structure). *A smooth structure on a topological n-manifold M is a maximal smooth atlas.*

Definition (Smooth Manifold (with Boundary)). *A smooth manifold is a pair $(M, \mathscr{A})$, where M is a topological manifold (with boundary) and $\mathscr{A}$ is a smooth structure on M.*

1.2 Problems

Problem (1-1). *Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the line with two origins.)*

Solution (1-1). *It is obvious that M is second-countable with basis of $\mathbb{R}^2$ intersected with M as M's basis. It is locally homeomorphic to $\mathbb{R}^1$. It is not Hausdorff, because $(0, -1)$ and $(0, 1)$ can not be seperated.*

Problem (1-2). *Show that a disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.*

Solution (1-2). *Let $M = \bigcup_{i \in I} \mathbb{R}_i$, where I is the uncountable index set and each $\mathbb{R}_i$ is a copy of $\mathbb{R}^1$. The open sets of M is the union of the open sets of $\mathbb{R}_i$. Thus, M is locally homeomorphic to $\mathbb{R}$ and Hausdorff. (The disjoint union of Hausdorff spaces is also Hausdorff.) However, there are uncountable number of basis for M. (A manifold should have countably many components, which is a natural result of the second-countable condition.)*

Problem (1-3). *A topological space is said to be σ-compact if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

Solution (1-3). *Assume M is a locally Euclidean Hausdorff space.*
"$\Longrightarrow$". By the definition of manifold, we only need to show that M is second-countable. Let $M = \bigcup_{i \in I} C_i$, where I is a countable index set and each C_i is a compact set. As C_i compact, each C_i is the union of finitely many open sets. Thus, M is a union of countably many open sets, which making it second-countable.
"$\Longleftarrow$". If M is second-countable, $M = \bigcup_{i \in I} B_i = \bigcup_i \bar{B}_i$, where I is a countable index set and B_i is the open ball. As each $\bar{B}_i$ is a compact set, M is σ-compact naturally.

Problem (1-4)**.** *Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.*
(a) Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.
(b) Give an example to show that the converse to (a) may be false.
(c) Now assume that the sets in $\mathcal{U}$ are precompact in M, and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Solution (1-4)**.**
(a) For each $p \in M$, we have $p \in U \in \mathcal{U}$, where U is some open set. As the neighborhood U of p intersects finitely many others, $\mathcal{U}$ is locally finite.
(b) If for each $p \in M$, p intersects finitely many sets in $\mathcal{U}$. I can not find a converse to (a). There is one counterexample provided by others, where $M = \mathbb{R}$ and $\mathcal{U} = \{\mathbb{R}\} \bigcup \{(i - 0.5, i + 1.5) : i \in \mathbb{Z}\}$.
(c) Let $U \in \mathcal{U}$ be an open set in M. As M is a topological manifold, $\bar{U}$ is covered by countably many open sets (just choose basis) of M. Name $\bar{U} \subset \bigcup V_i$. Without loss of generality, we may assume each open set V_i is the neighborhood of some point $p \in U$ that intersects finitely in $\mathcal{U}$. Name them $V_{ij} \in \mathcal{U}$ for each V_i. As U is precompact, the closure of U is compact. Thus, we have finitely many $\{V_i\}$ to cover $\bar{U}$. Thus, U at most intersects $\bigcup \{V_{ij}\}$, which is finite by finite in $\mathcal{U}$.

Problem (1-5)**.** *Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]*

Solution (1-5)**.**
"$\Longrightarrow$". If M is second-countable, it is a manifold. Thus, M is paracompact and has countably many connected components.
A topological basis is a subset B of a set T in which all other open sets can be written as unions or finite intersections of B.
"$\Longleftarrow$". A topological basis is a subset in which all other open sets can be written as unions or finite intersections of it. Assume N is some connected component of M. We only need to show that N is second-countable. For any open set $U \subset N$, there is a refinement $\mathcal{U}_U$ of $\{U, N - U\}$, which is locally finite. And $\mathcal{U}_U$ is at most countable, because we can use the property of Hausdorff to construct the open set step by step. Gather those countable open sets in $\mathcal{U}_U$ for each U (which is also at most countable by the property of Hausdorff). Then, we get countably many basis for N.

Note that, the construction in another proof is also interesting.

Problem (1-6)**.** *Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0, F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]*

Solution (1-6)**.** *Let (U, ϕ) be a smooth structure of M and for any $s > 0, F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $\mathbb{B}^n$ to itself. As $\phi(U)$ is homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^n$ is homeomorphic to $\mathbb{B}^n$, with loss of generality, we assume that $\phi(U) = \mathbb{B}^n$. Thus, $(U, F_s \circ \phi)$ is also a smooth structure of M. As s has uncountably many distinct one, there are uncountably many distinct smooth structure of M.*

Note that, the construction is not right. The right one is in the answer.

Problem (1-7)**.** *Let N denote the north pole $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \ldots, 0, -1)$. Define the stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by*

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

(a) For any $x \in \mathbb{S}^n \backslash \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called stereographic projection from the south pole.)

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\tilde{\sigma}$ are called stereographic coordinates.)

(d) Show that this smooth structure is the same as the one defined in Example 1.31. (Used on pp. 201, 269, 301, 345, 347, 450.)

Solution (1-7)**.**
(a) Let $\frac{x^1}{y^1} = \cdots = \frac{x^n}{y^n} = \frac{x^{n+1}-1}{y^{n+1}-1} = t$, and $x^{n+1} = 0$. Then we get the result. For $\widetilde{\sigma}(x)$, it is the same.
(b) For $(0, \cdots, 0, 1)$ and $(y^1, \cdots, y^n, 0)$, we have their parameterized point in the line, $(ty^1, \cdots, ty^n, 1-t)$. If the point is on $\mathbb{S}^n$, we get $t = \frac{2}{1+|y|^2}$. Thus, we get the inverse of σ.
For (c) and (d), the proof is trivial.

Solution (1-8)**.**
First, we prove the if and only if part.
"$\Longrightarrow$". Assume θ is an angle function: $\mathbb{S}^1 \to \mathbb{R}$. Because $e^{i\theta(z)} = z$, we get that θ is injective. As θ is continuous and $\mathbb{S}^1$ is connected and compact, $Im(\theta)$ should also be connected and compact in $\mathbb{R}$, which we take it to be $[a, b]$. Thus, $\mathbb{S}^1$ is homeomorphic to $[a, b]$ (injective, surjective, and continuous map). However, it is obvious that these two space is not homeomorphic as their fundamental group is different. Thus, we get a contradiction.
"$\Longleftarrow$". If $U \subsetneqq \mathbb{S}^1$, let $\widetilde{\theta}$ be the common "angle function" with $\widetilde{\theta}(1, 0) = 0$, $\widetilde{\theta}(0, 1) = 0.5\pi$, etc. We can take $\theta = \widetilde{\theta}|_U$. Second, the smoothness of $\theta = \widetilde{\theta}|_U$ is ovbious.

Solution (1-9)**.** *$\mathbb{C}P^n$ is homeomorphic to $\mathbb{S}^n$ in $\mathbb{C}^{n+1}$, by mapping each ray to a point in the sphere. Thus, it is a topological $2n$-manifold. The smooth structure is the reverse of the mapping composed with the smooth structure of $\mathbb{S}^n$.*

Note that I do not what I was saying, which is totally wrong!

Solution (1-10)**.** *The main idea is to construct the basis of S. As S intersects with Q trivially, the projection map $\pi|_P(S)$ is isomorphic to S. There exists $b_i \in Q$, such that $e_i + b_i \in S$, for $1 \leq i \leq k$. Gathering these b_i as columns, we get matrix B. The uniqueness is easily proved.*

Solution (1-11)**.** *The proof of boundary point and interior point is obvious. The standard smooth structure on $\mathbb{B}^n$ is the identity map on each open set. The problem is to construct a smooth structure on the boundary that is also compatible with interior smooth structure. The intuition is to map each $U_i = \{(x_1, \cdots, x_n),\ x_i > 0\} \bigcap \bar{\mathbb{B}}^n$ to $V_i = \{(x_1, \cdots, x_n),\ x_i < 0\} \bigcap \mathbb{B}^n$.*

Solution (1-12)**.** *Let M be a smooth m-manifold and N a smooth n-manifold with boundary. We prove that $M \times N$ a smooth $(m+n)$-manifold with boundary $M \times \partial N$. First, $M \times N$ is Hausdorff and second-countable because M and N are. Second, $\partial(M \times N) = M \times \partial N$ because $(x, y) \in \partial(M \times N)$ must have the form that $y \in \partial N$.*

Note that there is classification of smooth 1-manifold (with boundary), which is illustrated in https://www.math.tecnico.ulisboa.pt/~ggranja/TD/08/classif1manifs.pdf. Any smooth 1-manifold without boundary is diffeomorphic to $\mathbb{R}$ or S^1.

2 Smooth Maps

2.1 Concepts

Definition (Smooth Function)**.** *Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. f is a smooth function if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subset \mathbb{R}^n$.*

Definition (Smooth Map)**.** *Suppose M is a smooth m-manifold, N is a smooth n-manifold, and $F : M \to N$ is any map. F is a smooth map if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and a smooth chart (V, ψ) for N whose domain contains $F(p)$, such that the composite function $\psi \circ F \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subset \mathbb{R}^m$.*

Definition (Diffeomorphism)**.** *If M and N are smooth manifolds with or without boundary, a diffeomorphism from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse.*

Definition (Partition of Unity)**.** *Suppose M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M, indexed by a set A. A partition of unity subordinate to $\boldsymbol{X}$ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:*

1. *$0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.*
2. *$\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.*
3. *The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite, meaning that every point has a neighborhood that intersects supp ψ_α for only finitely many values of α.*
4. *$\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.*

Definition (Bump Function)**.** *If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a bump function for A supported in U if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subseteq U$.*

Definition (Exhaustion Function)**.** *If M is a topological space, an exhaustion function for M is a continuous function $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ (called a sublevel set of f) is compact for each $c \in \mathbb{R}$.*

2.2 Problems

Problem (2-1)**.** *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\big(U \cap f^{-1}(V)\big)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Solution (2-1)**.** *As f is smooth away from $x = 0$, we only think about the chart of (U, ϕ) where $0 \in U$. Define (V, ψ): $V = (1-\epsilon, 1+\epsilon)$, $\psi(x) = x$ and $U = (-\epsilon, \epsilon)$, $\phi(x) = x$, which are smooth coordinate charts. Then, we have $\phi(U \bigcap f^{-1}(V)) = \phi([0, \epsilon)) = [0, \epsilon)$. Thus, $\psi \circ f \phi^{-1}(x) = x$ in $\phi(U \bigcap f^{-1}(V))$, which is smooth. As f is not continuous, it is not smooth in the sense we have defined in this chapter. The main difference between the definition of f (with (U, ϕ) and (V, ψ)) and the definition of smooth map in this chapter, is that the latter one requires that $f(U) \subset V$.*

2-2. Prove Proposition 2.12 (smoothness of maps into product manifolds).

Problem (2-3)**.** *For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.*
(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the nth power map for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map $\alpha(x) = -x$.
(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\left\{(w, z) : |w|^2 + |z|^2 = 1\right\}$ of $\mathbb{C}^2$.

Solution (2-3)**.** *The main computation in this problem is to take the coordinate representations (U_i, ϕ_i) of each map F, compute $\phi_j \circ F \circ \phi_i$, and show that is a smooth function. For general S^n, we have stereographic chart. For $S^1 \in \mathbb{C}$, we have angle coordinate chart, where the map sends e^{ix} to x.*

Problem (2-4)**.** *Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.*

Solution (2-4)**.** *First, we give $\bar{\mathbb{B}}^n$ a smooth structure $\{(\mathbb{B}, Id)\} \bigcup \{(U_i^{\pm}, \phi_i^{\pm}) : i \in [n]\}$, where $U_i^+ = x \in \mathbb{R}^n : x^i > 0$, $\phi_i^+ : U_i^+ \to B_i^-$ and $U_i^- = x \in \mathbb{R}^n : x^i < 0$, $\phi_i^- : U_i^- \to B_i^+$, as defined in Problem 1-11. Then, the smoothness of the inclusion map is a straightforward calculation.*

Problem (2-5)**.** *Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.*
(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.
(b) Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3 .

Solution (2-5)**.** *In Example 1.23, the smooth structure is $(\mathbb{R}, \phi(x) = x^3)$.*
(a) $\phi \circ f \circ Id^{-1}(x) = f^3(x)$ is smooth.
(b) $Id \circ f \circ \phi^{-1}(x) = f(x^{\frac{1}{3}})$. Consider the Taylor series of f, the existence of which is guranteed by smoothness. In my view, $f(x^{\frac{1}{3}})$ is smooth if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Problem (2-6)**.** *Let $P : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{k+1} \backslash \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}, P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \backslash \{0\}$ and $x \in \mathbb{R}^{n+1} \backslash \{0\}$. (Such a function is said to be homogeneous of degree d.) Show that the map $\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.*

Solution (2-6)**.** *For $x = \lambda y$, $\widetilde{P}([x]) = \widetilde{P}([\lambda y]) = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \widetilde{P}([y])$. Thus, $\widetilde{P}([x]) = [P(x)]$ is well defined. Apply the smooth structure of $\mathbb{RP}^n$ and $\mathbb{RP}^k$, we get the smoothness of $\widetilde{P}$.*

Problem (2-7)**.** *Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional. [Hint: show that if $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]*

Solution (2-7)**.** *As $n \geq 1$, there is a nonempty open set U of M (we can choose one basis). For any given $k\mathbb{Z}^+$, choose k disjoint points $p_i, 1 \leq k \leq k$ from U and get k open set U_i such that $p_i \in U_i$ and U_i are disjoint from each other (the existence is guranteed by Hausdorff property). Let f_i be an element of partition of unity in $K_i \subset U_i$, where K_i is a closed set. Then $f_1, \ldots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, which is linearly independent obviously.*

Problem (2-8)**.** *Define $F : \mathbb{R}^n \to \mathbb{RP}^n$ by $F\left(x^1, \ldots, x^n\right) = \left[x^1, \ldots, x^n, 1\right]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G : \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G\left(z^1, \ldots, z^n\right) = \left[z^1, \ldots, z^n, 1\right]$ (see Problem 1-9).*

Solution (2-8)**.** *It is obvious that $F(\mathbb{R}^n) = \{\left[x^1, \ldots, x^n, x^{n+1}\right] : x^{n+1} > 0\} \triangleq U$. As U is smooth chart for $\mathbb{RP}^n$, it is open. Then, we need to show that U is dense in $\mathbb{RP}^n$. For any point $\widetilde{p} \in \mathbb{RP}^n / U$, set $\widetilde{p} = [p^1, \ldots, p^{n+1}] = \pi(p^1, \ldots, p^{n+1})$, where π is the smooth chart for $\mathbb{RP}^n$. There is a neighborhood V of $p = (p^1, \ldots, p^{n+1})$, and a point $q \in V$ such that $q^{n+1} > 0$. Then $\widetilde{q} \in U \bigcap \pi(V)$. Thus, U is dense.*

Problem (2-9)**.** *Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\widetilde{p} : \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1 -dimensional complex projective space and $G : \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8: (Used on p. 465.)*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \end{array}$$

Solution (2-9)**.** *Assume $p(z) = \sum_{i=0}^n a_i z^n$. For $(z_1, z_2) \in U_2^+$, let $\widetilde{p}([z_1, z_2]) = [\sum_{i=0}^n a_i z_1^i z_2^{n-i}]$ (for $(z_1, z_2) \in U_2^-$, it is the same). Then, it is a straightforward computation that the diagram commutes. The uniqueness is guaranteed by the dense property of G and the continuity of $\widetilde{p}$.*

Problem (2-10)**.** *For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.*
(a) Show that F^* is a linear map.
(b) Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*\left(C^\infty(N)\right) \subseteq C^\infty(M)$.
(c) Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.
Remark: this result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, some authors define a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties. (Used on p. 75.)

Solution (2-10)**.**
(a) $F^*(af + bg) = (af + bg) \circ F = af \circ F + bg \circ F = aF^*(f) + bF^*(g)$.
(b) The "$\Longrightarrow$" part is obvious, as $f \circ F$ is smooth if both f and F are. To prove the "$\Longleftarrow$" part, we need to prove that $\psi \circ F \circ \phi^{-1}$ is smooth for each chart (U, ϕ) in M and (V, ψ) in N. As ϕ^{-1} is smooth, a straightforward idea is to set f_i for each component function of ψ and $\psi \circ F = (f_i \circ F)$. As $f_i \circ F$ is smooth by the assumption, the proof seems to be done. However, there is a problem that f_i is defined in M and ψ is defined in $V \subset M$. Thus, there should be some modifications. We can get a closed ball W in V, and get a partition of unity u which satisfies $u(x) = 1$ when $x \in W$ and $supp(u) \subset V$. Let $f_i = \psi_i u$, we are done.
(c) "$\Longrightarrow$". If F is a diffeomorphism, then F^* is invertible linear map ($(F^{-1})^*$ as the inverse), linear map between infine dimensional spaces. Thus, F^* is an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.
"$\Longleftarrow$". As F^* is an isomorphism from $C^\infty(N)$ to $C^\infty(M)$, we have $F^*\left(C^\infty(N)\right) = C^\infty(M)$ and $(F^{-1})^*\left(C^\infty(M)\right) = C^\infty(N)$. By the result of (b), F and F^{-1} are both smooth. Thus, F is a diffeomorphism.

Problem (2-11)**.** *Suppose V is a real vector space of dimension $n \geq 1$. Define the projectivization of V, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \backslash \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}\left(\mathbb{R}^n\right) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E\left[v^1, \ldots, v^n\right] = \left[v^i E_i\right]$ (where brackets denote equivalence classes) is a diffeomorphism. (Used on p.561.)*

Solution (2-11)**.** *Take the standard orthogonal basis of V and build the smooth structure as $\mathbb{RP}^{n-1}$.*

Note that the answer of this problem is not such easy.

Problem (2-12)**.** *State and prove an analogue of Problem 2-11 for complex vector spaces.*

Problem (2-13)**.** *Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.*

Solution (2-13)**.** *For any open cover $\mathcal{X}$ of M, from the definition of the partition of unity, there is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions, such that supp$(\psi_\alpha) \subset X_\alpha$ for each α. As supp$(\psi_\alpha) = \psi^{-1}((0, +\infty))$ and ψ is continuous, thus supp(ψ_α) is an open set. The property of intersecting finite open set for each point is also from the partition of unity. Thus, M is paracompact.*

Problem (2-14)**.** *Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.*

Solution (2-14)**.** *From Level Sets of Smooth Functions Theorem, there are two smooth nonnegative functions $g, h : M \to \mathbb{R}$ such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. Let $f = \frac{g}{g+h}$.*

3 Tangent Vectors

3.1 Concepts

Definition (Tangent Vector and Tangent Space). *Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map $v : C^{\infty}(M) \to \mathbb{R}$ is called a derivation (tangent vector) at p if it satisfies*

$$v(fg) = f(p)vg + g(p)vf \quad \text{for all } f, g \in C^{\infty}(M)$$

The set of all derivations of $C^{\infty}(M)$ at p, denoted by T_pM, is a vector space called the tangent space to M at p.

Definition (Differential). *If M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map*

$$dF_p : T_pM \to T_{F(p)}N,$$

called the differential of F at p, as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^{\infty}(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Definition (Tangent Bundle). *Given a smooth manifold M with or without boundary, the tangent bundle of M, denoted by TM, is the disjoint union of the tangent spaces at all points of M :*

$$TM = \coprod_{p \in M} T_pM.$$

Definition (Global Differential). *If M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map. By putting together the differentials of F at all points of M, we obtain a globally defined map between tangent bundles, called the global differential and denoted by $dF : TM \to TN$. This is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p.*

Definition (Velocity Vector). *let M be a smooth manifold with or without boundary. Given a smooth curve $\gamma : J \to M$ and $t_0 \in J$, the velocity of γ at t_0 is $\gamma'(t_0)$, to be the vector*

$$\gamma'(t_0) = d\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) \in T_{\gamma(t_0)}M,$$

where $\left. d/\,dt \right|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

3.2 Problems

Problem (3-1). *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.*

Solution (3-1). *For any p in M, choose smooth coordinate charts (U, ϕ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation $\hat{F} = \psi \circ F \circ \phi^{-1}$ and*

$$dF_p(\left. \frac{\partial}{\partial x^i} \right|_p) = \frac{\partial \hat{F}^j}{\partial x^i}(\phi(p)) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}$$

"$\Longleftarrow$". *F is constant* $\Rightarrow \hat{F} = \psi \circ F \circ \phi^{-1}$ *is a constant* $\Rightarrow \frac{\partial \hat{F}^j}{\partial x^i} \triangleq 0 \Rightarrow dF_p$ *is the zero map.*
"$\Longrightarrow$". *dF_p is the zero map.* $\Rightarrow \frac{\partial \hat{F}^j}{\partial x^i}(\phi(p)) \frac{\partial}{\partial y^j}\Big|_{F(p)} = 0$ *for all j* $\Rightarrow \frac{\partial \hat{F}^j}{\partial x^i}(\phi(p)) = 0$ *because* $\frac{\partial}{\partial y^j}\Big|_{F(p)}$ *is the basis of $TN_{F(p)}$, and p can be any point in U.* $\Rightarrow \frac{\partial \hat{F}^j}{\partial x^i} \equiv 0$ *in U* $\Rightarrow \hat{F} = \psi \circ F \circ \phi^{-1}$ *is a constant* $\Rightarrow F$ *is constant.*

Problem (3-4)**.** *Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.*

Solution (3-4)**.** *Choose a smooth structure of $\mathbb{S}^1$ with two smooth charts, $(U \triangleq \mathbb{S}^1/\{(1,0)\}, \phi)$ and $(V \triangleq \mathbb{S}^1/\{(-1,0)\}, \psi)$, where ϕ and ψ are both anticlockwise angle functions. Thus, for any point $p \in U \bigcap V$, $\frac{d}{d\phi}\Big|_p = \frac{d}{d\psi}\Big|_p$. Define a map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ by $F(p \in U, \frac{d}{d\phi}\Big|_p \in T_pU) = (p, \frac{d}{d\phi}\Big|_p)$ and $F(p \in V, \frac{d}{d\psi}\Big|_p \in T_pV) = (p, \frac{d}{d\psi}\Big|_p)$. By Gluing Lemma for Smooth Maps. F exists and is smooth for $T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$. Because F is diffeomorphic for U and V respectively, the proof is done.*

Problem (3-5)**.** *Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x,y) : \max(|x|,|y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF\left(\gamma'(t)\right)$ on the coordinate functions x and y.] (Used on p. 123.)*

Solution (3-5)**.**
(a) The homeomorphism could be construct by mapping each $\frac{1}{8}$ arc to $\frac{1}{8}$ side.
(b) If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism such that $F\left(\mathbb{S}^1\right) = K$. Consider a chart (U,ϕ) containing $(1,1) \in K$ and a chart (V,ψ) containing $F^{-1}((1,1)) \in \mathbb{S}^1$. Let γ be a smooth curve whose image lies in $\mathbb{S}^1$. Then, $F \circ \gamma$ is a smooth curve whose image lies in K. In a small neighborhood of $F^{-1}((1,1)) \in \mathbb{S}^1$ in V, $F \circ \gamma$ is constant at x-axis or y-axis. Thus, $dF\left(\gamma'(t)\right) = (F \circ \gamma)'(t) \equiv 0$ in a small neighborhood of $(1,1) \in K$ in U. This contradicts the assumption that $F \circ \gamma$ is a smooth curve whose image lies in K.

Problem (3-6)**.** *Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = \left(z^1, z^2\right) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = \left(e^{it}z^1, e^{it}z^2\right)$. Show that γ_z is a smooth curve whose velocity is never zero.*

Solution (3-6)**.** $\gamma_z'(t) = (ie^{it}z^1, ie^{it}z^2) = i\gamma_z(t)$. *As $\gamma_z(t) \in \mathbb{S}^3$ is never zero, then $\gamma_z'(t)$ is never zero.*

Note that the answer uses charts for $\mathbb{S}^3$, which is more formal.

Problem (3-7)**.** *Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathscr{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathscr{D}_pM \to T_pM$ by $(\Phi v)f = v\left([f]_p\right)$. Show that Φ is an isomorphism. (Used on p. 71.)*

Problem (3-8)**.** *Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. (Used on p.72.)*

4 Submersions, Immersions, and Embeddings

4.1 Concepts

Definition (Smooth Submersion). *If M and N are smooth manifolds with or without boundary, a smooth map $F : M \to N$ is called a smooth submersion if its differential is surjective at each point (or equivalently, if* $\operatorname{rank} F = \dim N$ *).*

Definition (Smooth Immersion). *If M and N are smooth manifolds with or without boundary, a smooth map $F : M \to N$ is called a smooth immersion if its differential is injective at each point (or equivalently, if* $\operatorname{rank} F = \dim M$ *).*

Definition (Smooth Embedding). *If M and N are smooth manifolds with or without boundary, a smooth embedding of M into N is a smooth immersion $F : M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.*

Definition (Section of Map). *If M and N are smooth manifolds with or without boundary, $\pi : M \to N$ is any continuous map, a section of π is a continuous right inverse for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$.*

Definition (Topological Immersion). *If X and Y are topological spaces, a continuous map $F : X \to Y$ is called a topological immersion if every point of X has a neighborhood U such that $F|_U$ is a topological embedding.*

Definition (Topological Submersion). *If X and Y are topological spaces, a continuous map $\pi : X \to Y$ is a topological submersion if every point of X is in the image of a (continuous) local section of π.*

Definition (Smooth Covering Map). *If E and M are connected smooth manifolds with or without boundary, a map $\pi : E \to M$ is called a smooth covering map if π is smooth and surjective, and each point in M has a neighborhood U such that each component of $\pi^{-1}(U)$ is mapped diffeomorphically onto U by π.*

4.2 Problems

Problem (4-1). *Use the inclusion map $\mathbb{H}_n \hookrightarrow \mathbb{R}_n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary. (Used on p. 80.)*

Theorem 4.5 (Inverse Function Theorem for Manifolds). Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Solution (4-1). *Let $M = \mathbb{H}_n$, $N = \mathbb{R}_n$, $F = \iota : \mathbb{H}_n \hookrightarrow \mathbb{R}_n$, and $p = 0 \in \mathbb{H}_n$. Then, $dF_p = Id$ is invertible. However, any connected neighborhoods U_0 of $p = 0$ and V_0 of $F(p) = 0$ is not diffeomorphic, as the formal one is manifold with nonempty boundary and the latter one is not.*

Problem (4-2). *Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \to N$ is smooth. Show that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \operatorname{Int} N$. (Used on pp .80,87.)*

Solution (4-2). *Assume $F(p) \in \partial N$, then there is a chart (V, ψ) for $F(p)$ such that $\psi(V)$ is diffeomorphic to $\mathbb{H}^n$ and $\psi(F(p)) = 0$. Assume (U, ϕ) is a chart for p. Let $(x_i)_{i\in[m]}$ be the coordinate of U and $(y_i)_{i\in[n]}$ be* *the coordinate of V. Then $F(q)^n \equiv 0$ for any $q \in U$. The differential of F at p is*

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \cdots & \frac{\partial F^n}{\partial x^m}(p) \end{pmatrix}$$

and $\frac{\partial F^n}{\partial x^i} \equiv 0$ for $i \in [m]$. Thus, dF_p is singular, which is a contradiction.

Problem (4-4)**.** *Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. (Used on pp .502,542.)*

Solution (4-4)**.** *The proof is finished in the learning of Point Set Topology.*

Problem (4-5)**.** *Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space, as defined in Problem 1-9. (a) Show that the quotient map $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ is a surjective smooth submersion. (b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$. (Used on pp. 172, 560.)*

Solution (4-5)**.**
(a) The surjection is by the definition of $\mathbb{CP}^n$. For any $p = (z_1, \ldots, z_n, z_{n+1}) \in U_{n+1}$, $\pi(p) = [z_1, \ldots, z_n, z_{n+1}]$. Let (U, ϕ_i) be a chart containing $\pi(p)$, $\phi_i \circ \pi \circ \mathrm{Id}^{-1}(z_1, \ldots, z_{n+1}) = (\frac{z_1}{z_{n+1}}, \ldots, 1, \ldots \frac{z_n}{z_{n+1}})$, which is smooth. For submersion, differentiate this map and we can get $\mathrm{rank}(dF_p) = 2n$. (b) This is a long way to learn detailly.

Problem (4-6)**.** *Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.*

Solution (4-6)**.** *By Properties of Smooth Submersions, F is smooth submersion $\to$ F is open map $\to$ $F(M)$ is an open set in $\mathbb{R}^k$. Besides, F is smooth $\to$ F is continuous, M is compact $\to$ $F(M)$ is compact in $\mathbb{R}^k$ $\to$ $F(M)$ is a bounded and closed set in $\mathbb{R}^k$. Thus, $F(M)$ can only be empty set, which is a contradiction.*

Problem (4-8)**.** *This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.*

Solution (4-8)**.** *π is smooth obviously. $\pi(\mathbb{R}, 1) = \mathbb{R}$ shows that π is surjective. $dF_{(x,y)} = (y, x) \to \mathrm{rank}\, dF_{(0,0)} = 0 < 1 \to F$ is not a submersion. If $F : \mathbb{R} \to P$ is smooth, then $F \circ \pi$ is smooth by composition. If $F \circ \pi$ is smooth, then $F(x) = F \circ \pi(x, 1)$ is also smooth.*

Problem (4-10)**.** *Show that the map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map. (Used on pp .550.)*

Solution (4-10)**.** *For $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{S}^n$, $q(x_1, \ldots, x_n, x_{n+1}) = q(-x_1, \ldots, -x_n, -x_{n+1}) = [x_1, \ldots, x_n, x_{n+1}]$. The map is smooth and surjective obviously. For each $p \in \mathbb{RP}^n$, it must be contained in some chart (V_i, ψ_i) of $\mathbb{RP}^n$. $V_i = \{[x_1, \ldots, x_n, x_{n+1}] : x_i \neq 0\}$ is diffeomorphic to $U_i^+ = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{S}^n : x_i > 0\}$ and $U_i^- = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{S}^n : x_i < 0\}$, which are equal to $q^{-1}(V_i)$ and mapped diffeomorphically onto V_i.*

Problem (4-12)**.** *Using the covering map $\varepsilon^2 : \mathbb{R}^2 \to \mathbb{T}^2$ (see Example 4.35), show that the immersion $X : \mathbb{R}^2 \to \mathbb{R}^3$ defined in Example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\widetilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$, and then show that $\widetilde{X}$ is a smooth embedding whose image is the given surface of revolution.*

Solution (4-12)**.** *For $X : \mathbb{R}^2 \to \mathbb{R}^3$, $X(u, v) = ((2 + \cos 2\pi u)\cos 2\pi v, (2 + \cos 2\pi u)\sin 2\pi v, \sin 2\pi u)$, it is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis (Fig. 4.1). X is a constant in each fiber of ε^2 as X has period of positive integer in two input.*

As ε^2 is surjective and smooth submersion (smooth covering map), using Theorem 4.30 (Passing Smoothly to the Quotient), for the immersion $X : \mathbb{R}^2 \to \mathbb{R}^3$, there is a unique smooth map $\widetilde{X} : \mathbb{T}^2 \to \mathbb{R}^3$ such that $X = \widetilde{X} \circ \varepsilon^2$. Thus, $\mathrm{Im}(\widetilde{X})$ is the given surface of revolution (doughnut-shaped surface).

Then, we need to prove that $\widetilde{X}$ is a smooth embedding. An intuition is to prove that $\widetilde{X}$ is an injective immersion and as $\mathbb{T}^2$ is compact we can use Proposition 4.22 to prove that $\widetilde{X}$ is an embedding. $dX_p = d\widetilde{X}_{\varepsilon^2(p)} \circ d\varepsilon^2_p$, X is an immersion and ε^2 is a locally diffeomorphism $\Rightarrow \mathrm{rank}(\widetilde{X}) = \mathrm{rank}(X) = 2$, ε^2 is surjective $\Rightarrow \widetilde{X}$ is an immersion.

To prove that $\widetilde{X}$ is injective, assume $p_1 = (e^{2\pi x^1 i}, e^{2\pi x^2 i}) \in \mathbb{T}^2$, $p_2 = (e^{2\pi y^1 i}, e^{2\pi y^2 i}) \in \mathbb{T}^2$ and $\widetilde{X}(p_1) = \widetilde{X}(p_2)$. Then

$$
\begin{aligned}
(2+\cos 2\pi x^1)\cos 2\pi x^2 &= (2+\cos 2\pi y^1)\cos 2\pi y^2 \\
(2+\cos 2\pi x^1)\sin 2\pi x^2 &= (2+\cos 2\pi y^1)\sin 2\pi y^2 \\
\sin 2\pi x^1 &= \sin 2\pi y^1
\end{aligned}
$$

To satisfy $\sin 2\pi x^1 = \sin 2\pi y^1$, we have two choices, (1) $2\pi x^1 = \pi - 2\pi y^1$ or (2) $2\pi x^1 = 2\pi y^1$. If the (2) is satisfied $\Rightarrow x^1 = x^2 \Rightarrow \cos 2\pi x^2 = \cos 2\pi y^2$ and $\sin 2\pi x^2 = \sin 2\pi y^2 \Rightarrow x^2 = y^2$. Then, we prove that $\widetilde{X}$ is injective. If (1) is satisfied, then $(2+\cos 2\pi x^1)\cos 2\pi x^2 = (2-\cos 2\pi x^1)\cos 2\pi y^2$ and $(2+\cos 2\pi x^1)\sin 2\pi x^2 = (2-\cos 2\pi x^1)\sin 2\pi y^2$, then $2\pi x^2 = 2\pi y^2 + \pi$ by the period of π for $\tan$. Then, we have $(2+\cos 2\pi x^1)\cos 2\pi x^2 = -(2-\cos 2\pi x^1)\cos 2\pi x^2 \Rightarrow \cos 2\pi x^2 = 0 \Rightarrow 2\pi x^2 = \frac{\pi}{2} + k\pi \Rightarrow 2\pi y^2 = 2\pi x^2 - \pi = \frac{\pi}{2} + k\pi = 2\pi x^2 \Rightarrow x^2 = y^2 \Rightarrow x^1 = y^1$. Thus, we also prove that $\widetilde{X}$ is injective.

Problem (4-13)**.** *Define a map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x,y,z) = \big(x^2 - y^2, xy, xz, yz\big)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.*

Solution (4-13)**.** *By Problem 4-10, $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ is a smooth covering map. At each fiber of π, F is a constant as F is quadratic homogeneous polynomial in each dimension. F is smooth obviously. Thus, by Theorem 4.30 (Passing Smoothly to the Quotient), we have there is a unique smooth map $\widetilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$.*

To prove that $\widetilde{F}$ is an embedding, I also want to prove that $\widetilde{F}$ is an injective immersion; then by the result that $\mathbb{RP}^2$ is a compact space ($\mathbb{S}^2$ is compact and its quotient is as well), we can use the Proposition 4.22 to prove that $\widetilde{F}$ is embedding. Calculate the matrix form of differential dF:

$$
dF = \begin{pmatrix} 2x & y & z & 0 \\ -2y & x & 0 & z \\ 0 & 0 & x & y \end{pmatrix}
$$

The determinant of sequential principal minor for dF is $2x(x^2+y^2) \neq 0$. Thus, F is immersion. Note that we can also prove that F is immersion by writing the formula of $\widetilde{F}$ explicitly,

$$
\widetilde{F}([x,y,z]) = \frac{1}{x^2+y^2+z^2}\left(x^2 - y^2, xy, xz, yz\right)
$$

Then, calculating the differential of $\widetilde{F}$ can show its immersion property.

As π is a locally diffeomorphism and surjective, $\widetilde{F}$ is immersion.

To prove that $\widetilde{F}$ is injective, assume we have (x,y,z) and (x',y',z') such that $x^2+y^2+z^2 = 1$, $x'^2 + y'^2 + z'^2 = 1$ and $\widetilde{F}([x,y,z]) = \widetilde{F}([x',y',z'])$. Then,

$$
x^2 - y^2 = x'^2 - y'^2,\ xy = x'y',\ xz = x'z',\ yz = y'z'
$$

According to those equations, we have $(x,y,z) = \pm(x',y',z')$. Thus, $[(x,y,z)] = [(x',y',z')]$ and $\widetilde{F}$ is injective.

5 Submanifolds

5.1 Concepts

Definition (Embedded Submanifold). *Suppose M is a smooth manifold with or without boundary. An embedded submanifold of M is a subset $S \subseteq M$ that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding.*

Definition (Immersed Submanifold). *Suppose M is a smooth manifold with or without boundary. An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth immersion.*

Definition (Embedded Topological Submanifold). *Suppose M is a topological manifold with or without boundary. An embedded topological submanifold of M is a subset $S \subseteq M$ that the inclusion map $S \hookrightarrow M$ is a topological embedding.*

Definition (Immersed Topological Submanifold). *Suppose M is a topological manifold with or without boundary. An immersed topological submanifold of M is a subset $S \subseteq M$ endowed with a topology such that it is a topological manifold and such that the inclusion map is a topological immersion.*

Definition (Properly Embedded). *An embedded submanifold $S \hookrightarrow M$ is said to be properly embedded if the inclusion is a proper map.*

Definition (Slice in $\mathbb{R}^n$). *If U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$, a k dimensional slice of U (or simply a k-slice) is any subset of the form*

$$S = \left\{ \left(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n\right) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n \right\}$$

for some constants $c^{k+1}, \ldots, c^n$. (When $k = n$, this just means $S = U$.)

Definition (Slice in Manifold). *Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a k-slice of U.*

Definition (Regular Point). *If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be a regular point of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective.*

Definition (Critical Point). *If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be a critical point of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is not surjective.*

Definition (Regular Value). *If $\Phi : M \to N$ is a smooth map, a point $c \in N$ is said to be a regular value of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point.*

Definition (Critical Value). *If $\Phi : M \to N$ is a smooth map, a point $c \in N$ is said to be a critical value of Φ if a point of the level set $\Phi^{-1}(c)$ is a critical point.*

Definition (Regular Level Set). *If $\Phi : M \to N$ is a smooth map, a level set $\Phi^{-1}(c)$ is called a regular level set if c is a regular value of Φ.*

5.2 Problems

Problem (5-1). *Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by*

$$\Phi(x, y, s, t) = \left(x^2 + y, x^2 + y^2 + s^2 + t^2 + y\right).$$

Show that $(0, 1)$ is a regular value of Φ, and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

Solution (5-1)**.** *Calculate the matrix of $d\Phi$,*

$$d\Phi = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}$$

The sequencial principal minor of $d\Phi$ has determinant $4xy$. If $y=0$ or $x=0$, then $x^2+y=0 \Rightarrow x=0$ and $y=0 \Rightarrow s^2+t^2=1 \Rightarrow \operatorname{rank}(d\Phi)=2$. If $x \neq 0$ and $y \neq 0$, then the sequencial principal minor of $d\Phi$ is >0 and $\operatorname{rank}(d\Phi)=2$. Thus, $(0,1)$ is a regular value of Φ.
For the level set $\Phi^{-1}(0,1) = \{(x,y,s,t) \in \mathbb{R}^4 : y=-x^2, x^4+s^2+t^2=1\}$. In fact, the image of $\Phi^{-1}(0,1)$ is not hard to image, if assuming s and t to be one variable we get a $\mathbb{S}^1$ embedded in $\mathbb{S}^2$. Construct a map $F^{\pm}$: $\Phi^{-1}(0,1) \to \mathbb{S}^2$, $F^{\pm}(x,y,s,t) = (\pm x^2, s, t)$, where $\pm$ is for different charts of $\mathbb{S}^2$. Then, we only need to prove that $F^{\pm}$ is diffeomorphic. Another idea is similar, let $F : \Phi^{-1}(0,1) \to S = \{(x,s,t) \in \mathbb{R}^3 : x^4+s^2+t^2=1\}$, $F(x,y,s,t)=(x,s,t)$. It is obvious that F is smooth and as y is uniquely determinated by x in $\Phi^{-1}(0,1)$, F is a diffeomorphism. Thus, the problem is converted to proving S is diffeomorphic to $\mathbb{S}^2$. We can build charts for S and prove it for each chart.

Note that, I do not know how to prove the diffeomorphism between S and $\mathbb{R}^2$. There is one answer that constructs a map $G : S \to \mathbb{S}^2$, $G(x,y,z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$ and G^{-1} with some computation. Then, it suffices to show that G has constant rank 2 (by Theorem 4.14 (Global Rank Theorem) we have G a diffeomorphism).

Problem (5-4)**.** *Show that the image of the curve $\beta : (-\pi,\pi) \to \mathbb{R}^2$ of Example 4.19 (The Figure-Eight Curve $t \to (\sin 2t, \sin t)$) is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.]*

Solution (5-4)**.** *Let $S = Im(\beta)$ be the Figure-Eight Curve with the subspace topology. If S has a smooth structure making it an embedded submanifold of $\mathbb{R}^2$, then the inclusion map $\iota : S \hookrightarrow \mathbb{R}^2$ is a smooth embedding. As $\operatorname{rank}(\beta) <= 1$, $\dim(S) \leq 1$. It is not 0-manifold as its topology is not discrete topology (inherited from $\mathbb{R}^2$). Thus, S is smooth 1-manifold. As smooth 1-manifold only have $\mathbb{R}$ and $\mathbb{S}$ up to diffeomorphism. As S is compact, it is not diffeomorphic to $\mathbb{R}$. As S has different fundamental group ($\mathbb{Z} \oplus \mathbb{Z}$) with $\mathbb{S}$ ($\mathbb{Z}$), they are not homeomorphic, and thus not diffeomorphic.*

Problem (5-5)**.** *Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that $\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]*

Solution (5-5)**.** *Let $S = Im(\gamma)$ be the Dense Curve of $\mathbb{T}^2$ with the subspace topology. If S has a smooth structure making it an embedded submanifold of $\mathbb{T}^2$, then the inlcusion map $\iota : S \hookrightarrow \mathbb{T}^2$ is a smooth embedding. It is not 0-manifold as its topology is not discrete topoloty (inherited from $\mathbb{T}^2$ and a single point in S is not open). If it is 1-manifold, it can not be diffeomorphic to $\mathbb{S}^1$ as it is not self-loop. It also can not be diffeomorphic to $\mathbb{R}$ as $\bar{\mathbb{R}} = \mathbb{R}$ is not self-loop while $\bar{S} = \mathbb{T}^2$. If S is smooth 2-manifold, S should cover some open set $U \times V$ of $\mathbb{T}^2$, including rational$\times$rational point, which is impossible. For some open set $U \times V$, we have $p = (e^{2\pi ti}, e^{2\pi\alpha ti}) = (e^{2\pi pi}, e^{2\pi qi})$ for some $p, q \in \mathbb{Q}$, which means that $(p+m)\alpha = (q+n)$ for some $m, n \in \mathbb{Z}$. It contradicts that α is an irrational.*

Note that to prove S is not 1-manifold, another idea is that $\mathbb{R}$ and $\mathbb{S}^1$ are both locally path-connected while S is not.

Problem (5-6)**.** *Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M :*

$$UM = \{(x,v) \in T\mathbb{R}^n : x \in M, v \in T_xM, |v| = 1\}.$$

*It is called the unit tangent bundle of **M**. Prove that UM is an embedded ($2m-1$)-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. (Used on p. 147.)*

Solution (5-6)**.** *First, $TM \subseteq T\mathbb{R}^n$ is an embedded 2m-dimensional submanifold, because we can use Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds) to construct the coordinate $(x_1, \ldots, x_m, 0, \ldots, 0)$ for M and thus $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}, 0, \ldots, 0)$ for T_pM. As U_pM is embedded $m-1$-submanifold for T_pM, UM is an embedded $2m-1$-dimensional submanifold for $T\mathbb{R}^n$.*

Problem (5-7)**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.*

Solution (5-7)**.** *$dF(x, y) = (3x^2 + y, 3y^2 + x)$. Its rank is equal to 1 when $(x, y) \in \mathbb{R}/\{(0, 0), (-\frac{1}{3}, -\frac{1}{3})\} = A$. By Theorem 5.12 (Constant-Rank Level Set Theorem), the level sets of F in $F(A)$ are all embedded 1-submanifolds of $\mathbb{R}^2$.*

For $F^{-1}(F(-\frac{1}{3}, -\frac{1}{3})) = F^{-1}(\frac{1}{27})$, we need to solve $x^3 + xy + y^3 = \frac{1}{27}$. By the method of undetermined coefficients, we have

$$x^3 + xy + y^3 - \frac{1}{27} = \left(x + y - \frac{1}{3}\right)\left(y^2 - \left(x - \frac{1}{3}\right)y + \left(x^2 + \frac{1}{3}x + \frac{1}{9}\right)\right)$$

Thus, $F^{-1}(\frac{1}{27}) = \{(x, y) \in \mathbb{R}^2 : x + y - \frac{1}{3} = 0\} \bigcup \{(-\frac{1}{3}, -\frac{1}{3})\}$. It is an embedded submanifold with 0-manifold and 1-manifold.

For $F^{-1}(F(0, 0)) = F^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x^3 + xy + y^3 = 0\}$, it is the folium of Descartes and not an embedded 1-manifold.

Problem (5-8)**.** *Suppose M is a smooth n-manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \backslash B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$. (Used on p.225.)*

Solution (5-8)**.** *It is obvious, as $\partial(M\backslash B)$ and ∂B is diffeomorphic to $\mathbb{S}^{n-1}$.*

6 Sard's Theorem

6.1 Concepts

Definition (Normal Space)**.** *Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold. For each $x \in M$, we define the normal space to M at x to be the $(n-m)$-dimensional subspace $N_xM \subseteq T_x\mathbb{R}^n$ consisting of all vectors that are orthogonal to T_xM with respect to the Euclidean dot product.*

Definition (Normal Bundle)**.** *Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold. The normal bundle of M, denoted by NM, is the subset of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ consisting of vectors that are normal to M :*

$$NM = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M, v \in N_xM\}.$$

Definition (Tubular Neighborhood)**.** *A tubular neighborhood of M is a neighborhood U of M in $\mathbb{R}^n$ that is the diffeomorphic image under E of an open subset $V \subseteq NM$ of the form*

$$V = \{(x, v) \in NM : |v| < \delta(x)\},$$

for some positive continuous function $\delta : M \to \mathbb{R}$.

Definition (Smooth Homotopy)**.** *If N and M are two smooth manifolds with or without boundary, a homotopy $H : N \times I \to M$ is called a smooth homotopy if it is also a smooth map, in the sense that it extends to a smooth map on some neighborhood of $N \times I$ in $N \times \mathbb{R}$.*

Definition (Smooth Homotopic)**.** *Two maps are said to be smoothly homotopic if there is a smooth homotopy between them.*

Definition (Transversality)**.**

Two submanifolds intersect transversely. Suppose M is a smooth manifold. Two embedded submanifolds $S, S' \subseteq M$ are said to intersect transversely if for each $p \in S \cap S'$, the tangent spaces T_pS and T_pS' together span T_pM (where we consider T_pS and T_pS' as subspaces of T_pM).

Map transverse to embedded submanifold. If $F : N \to M$ is a smooth map and $S \subseteq M$ is an embedded submanifold, we say that F is transverse to S if for every $x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ and $dF_x(T_xN)$ together span $T_{F(x)}M$.

6.2 Problems

7 Lie Groups

7.1 Concepts

7.2 Problems

8 Vector Fields

8.1 Concepts

8.2 Problems

9 Integral Curves and Flows

9.1 Concepts

9.2 Problems

10 Vector Bundles

10.1 Concepts

10.2 Problems