A Note of the Chromatic Number of Kneser Graphs

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This is a note of the chromatic number of Kneser graphs.

Pre-requisites of this note include a knowledge of the basic concepts of linear algebra, algebraic topology.

Contents

1 Introduction

In 1956, Kneser conjectured the chromatic number of the Kneser graph $\chi(KG(n,k)) = n-2k+2$ [8]. In 1978, Lovász proved this conjecture with topological methods [9]. At the same year, Bárány gave a simple proof [2], using the Borsuk–Ulam theorem [3] and a lemma of Gale [6]. In 2002, Greene simplified the proof of Bárány's without using Gale's lemma [7].

In 1976, Stahl conjectured the m-th multichromatic number of the Kneser graph $\chi_m(\mathrm{KG}(n,k)) = \left\lceil \frac{m}{k} \right\rceil (n-2k) + 2m$ [11].

In 2012, Meunier conjectured the chromatic number of the s-stable Kneser graph $\chi(\mathrm{KG}(n,k)_s) = n-sk+s$ [10]. In 2015, Chen proved this conjecture with a generalization to m-th multichromatic number of the s-stable Kneser graph $\chi_m(\mathrm{KG}(n,k)_s) = n-sk+sm$ [4]. In 2016, Alishahi and Hajiabolhassan proved a generalization of Gale's lemma and presented another proof of Chen's result [1].

2 Definition

Definition 1 (Kneser graph). *For $n \geq 2k$, the Kneser graph* $\mathrm{KG}(n,k)$ *is a graph whose vertex set consists of all k-subsets of $[n]$ and two vertices are adjacent if their corresponding k sets are disjoint.*

Definition 2 (Hemisphere $H(x)$). *For an $x \in S^d, H(x)$ is the open hemisphere centered at x, i.e. $H(x) = \{y \in S^d : \langle x, y\rangle > 0\}$.*

Definition 3 (s-stable). *For a positive integer s, a subset A of $[n]$ is said to be an s-stable subset if $s \leq |i-j| \leq n-s$ for each $i \neq j \in A$. The family of all s-stable k-subsets of $[n]$ is denoted by $\binom{[n]}{k}_s$*

Definition 4 (Length of the longest alternating subsequence). *For an $X = (x_1, \ldots, x_n) \in \{+,-,0\}^n$, an alternating subsequence of X is a subsequence of nonzero terms of X such that each of its two consecutive members have different signs. In other words, $x_{j_1}, \ldots, x_{j_m}$ $(1 \leq j_1 < \cdots < j_m \leq n)$ is an alternating subsequence of X if $x_{j_i} \neq 0$ for each $i \in [m]$ and $x_{j_i} \neq x_{j_{i+1}}$ for $i = 1, \ldots, m-1$. The length of the longest alternating subsequence of X is denoted by* $\mathrm{alt}(X)$.

Definition 5 (Signed-power set). *Let V be a nonempty finite set of size n. The signed-power set of V, denoted by $P_s(V)$, is defined as follows:*

$$P_s(V) = \{(A,B) : A, B \subseteq V, A \cap B = \varnothing\}$$

Definition 6 (Signed-increasing property). *A signed-increasing property $\mathcal{P}$, is a superset-closed family $\mathcal{P} \subseteq P_s(V)$, i.e. for any $F_1 \in \mathcal{P}$, if $F_1 \subseteq F_2 \in P_s(V)$, then $F_2 \in \mathcal{P}$.*

Definition 7 (X^+, X^-).

$$X^+ = \{j : x_j = +\} \quad \textit{and} \quad X^- = \{j : x_j = -\}$$

Definition 8 (Z_x^+, Z_x^-). *Let $d \geq 0$ be an integer, S^d be the d-dimensional sphere, and $Z \subset S^d$ be a finite set. For an $x \in S^d$, define $Z_x = (Z_x^+, Z_x^-) \in P_s(Z)$ where $Z_x^+ = H(x) \cap Z$ and $Z_x^- = H(-x) \cap Z$.*

Definition 9 (X_σ). *For any bijection $\sigma : [n] \to V$, $X_\sigma = (\sigma(X^+), \sigma(X^-))$ is an identification between $\{+,-,0\}^n$ and $P_s(V)$, where*

$$X^+ = \{j : x_j = +\} \quad \textit{and} \quad X^- = \{j : x_j = -\}$$

Definition 10 ($\mathrm{alt}(\mathcal{P}, \sigma)$). *Let $\sigma : [n] \to V$ be a bijection and $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property. Define*

$$\mathrm{alt}(\mathcal{P}, \sigma) = \max\{\mathrm{alt}(X) : X \in \{+,-,0\}^n \textit{ with } X_\sigma \notin \mathcal{P}\}$$

Definition 11 ($\mathrm{alt}(\mathcal{P})$).

$$\mathrm{alt}(\mathcal{P}) = \min\{\mathrm{alt}(\mathcal{P}, \sigma) : \sigma : [n] \to V \textit{ is a bijection }\}$$

Definition 12 (Chromatic number of hypergraph $\chi(\mathcal{H})$). *A t-coloring of a hypergraph $\mathcal{H}$ is a map $c : V(\mathcal{H}) \to [t]$ such that for no edge $e \in E(\mathcal{H})$, we have $|c(e)| = 1$. The chromatic number of $\mathcal{H}$ is the minimum possible t admitting a t-coloring, denoted by $\chi(\mathcal{H})$.*

Definition 13 (Kneser graph of hypergraph). *For a hypergraph $\mathcal{H}$, the Kneser graph of $\mathcal{H}$ is a graph whose vertex set is $E(\mathcal{H})$ and two vertices are adjacent if their corresponding edges are vertex disjoint, denoted by* $\mathrm{KG}(\mathcal{H})$.

Definition 14 (Kneser representation of graph). *For any graph G, the hypergraph $\mathcal{H}$, for which G and* $\mathrm{KG}(\mathcal{H})$ *is isomorphic, is called a Kneser representation of G.*

Definition 15 (Colorability defect of hypergraph)**.** *The colorability defect of a hypergraph* $\mathcal{H}$*, is the minimum number of vertices that should be excluded so that the induced subhypergraph on the remaining vertices is 2-colorable, denoted by* $\mathrm{cd}(\mathcal{H})$*.*

Definition 16 ($\mathrm{alt}(\mathcal{H},\sigma)$, $\mathrm{salt}(\mathcal{H},\sigma)$)**.** *Let* $\mathcal{H}=(V,E)$ *be a hypergraph and* $\sigma:[n]\to V(\mathcal{H})$ *be a bijection. Define*

$$\mathrm{alt}(\mathcal{H},\sigma)=\max\left\{\mathrm{alt}(X): X\in\{+,-,0\}^n \textit{ s.t. } \max\left(\left|E\left(\mathcal{H}\left[\sigma\left(X^+\right)\right]\right)\right|,\left|E\left(\mathcal{H}\left[\sigma\left(X^-\right)\right]\right)\right|\right)=0\right\}$$

and

$$\mathrm{salt}(\mathcal{H},\sigma)=\max\left\{\mathrm{alt}(X): X\in\{+,-,0\}^n \textit{ s.t. } \min\left(\left|E\left(\mathcal{H}\left[\sigma\left(X^+\right)\right]\right)\right|,\left|E\left(\mathcal{H}\left[\sigma\left(X^-\right)\right]\right)\right|\right)=0\right\}.$$

In other words, $\mathrm{alt}(\mathcal{H},\sigma)$ (resp. $\mathrm{salt}(\mathcal{H},\sigma)$) is the maximum possible $\mathrm{alt}(X)$, where $X\in\{+,-,0\}^n$, such that each of (resp. at least one of) $\sigma\left(X^+\right)$ and $\sigma\left(X^+\right)$ contains no edge of $\mathcal{H}$.

Definition 17 ($\mathrm{alt}(\mathcal{H})$, $\mathrm{salt}(\mathcal{H})$)**.**

$$\mathrm{alt}(\mathcal{H})=\min_{\sigma}\mathrm{alt}(\mathcal{H},\sigma)\quad \textit{and}\quad \mathrm{salt}(\mathcal{H})=\min_{\sigma}\mathrm{salt}(\mathcal{H},\sigma)$$

where the minimum is taken over all bijections $\sigma:[n]\to V(\mathcal{H})$*.*

Definition 18 ($\mathbb{Z}_2$-space)**.** *A* $\mathbb{Z}_2$*-space is a pair* (T,v)*, where* T *is a topological space and* v *is an involution, i.e.* $v:T\to T$ *is a continuous map such that* v^2 *is the identity map.*

Definition 19 (Free $\mathbb{Z}_2$-space)**.** *The* $\mathbb{Z}_2$*-space* (T,v) *is called free if there is no* $x\in T$ *such that* $v(x)=x$*.*

Definition 20 ($\mathbb{Z}_2$-map)**.** *For two* $\mathbb{Z}_2$*-spaces* (T_1,v_1) *and* (T_2,v_2)*, a continuous map* $f:T_1\to T_2$ *is called a* $\mathbb{Z}_2$*-map if* $f\circ v_1=v_2\circ f$*. The existence of such a map is denoted by* $(T_1,v_1)\xrightarrow{\mathbb{Z}_2}(T_2,v_2)$*.*

Definition 21 ($\mathbb{Z}_2$-index, $\mathbb{Z}_2$-coindex)**.** *For a* $\mathbb{Z}_2$*-space* (T,v)*, define the* $\mathbb{Z}_2$*-index and* $\mathbb{Z}_2$*-coindex of* (T,v)*, respectively, as*

$$\mathrm{ind}(T,v)=\min\left\{d\geq 0 (T,v)\xrightarrow{\mathbb{Z}_2}\left(S^d,-\right)\right\}$$

and

$$\mathrm{coind}(T,v)=\max\left\{d\geq 0 \left(S^d,-\right)\xrightarrow{\mathbb{Z}_2}(T,v)\right\}$$

If for any $d\geq 0$*, there is no* $(T,v)\xrightarrow{\mathbb{Z}_2}\left(S^d,-\right)$*, then we set ind* $(T,v)=\infty$*. Also, if* (T,v) *is not free, then ind* $(T,v)=\mathrm{coind}(T,v)=\infty$*.*

Note that if $T_1\xrightarrow{\mathbb{Z}_2}T_2$, then $\mathrm{ind}\,(T_1)\leq\mathrm{ind}\,(T_2)$ and $\mathrm{coind}\,(T_1)\leq\mathrm{coind}\,(T_2)$.

Definition 22 ($\mathbb{Z}_2$ equivalent)**.** *Two* $\mathbb{Z}_2$*-spaces* T_1 *and* T_2 *are* $\mathbb{Z}_2$ *equivalent, denoted by* $T_1\xrightarrow{\mathbb{Z}_2}T_2$*, if* $T_1\xrightarrow{\mathbb{Z}_2}T_2$ *and* $T_2\xrightarrow{\mathbb{Z}_2}T_1$*.*

Definition 23 (Abstract simplicial complex)**.** *An abstract simplicial complex is a pair* $L=(V,K)$*, where* V *(the vertex set of* L *) is a set and* $K\subseteq 2^V$ *(the set of simplices of* L *) is a hereditary collection of subsets of* V*, i.e. if* $A\in K$ *and* $B\subseteq A$*, then* $B\in K$*. Any set* $A\in K$ *is called a simplex of* L*.*

The geometric realization of an abstract simplicial complex L is denoted by $\|L\|$.

Definition 24 (Simplicial map)**.** *For two abstract simplicial complexes* $L_1=(V_1,K_1)$ *and* $L_2=(V_2,K_2)$*, a simplicial map* $f:L_1\to L_2$ *is a map from* V_1 *to* V_2 *preserving the simplices, i.e. if* $A\in K_1$*, then* $f(A)\in K_2$*.*

Definition 25 (Simplicial involution)**.** *A simplicial involution is a simplicial map $v : L \to L$ such that v^2 is the identity map.*

Definition 26 (Simplicial $\mathbb{Z}_2$-complex)**.** *A simplicial $\mathbb{Z}_2$-complex is a pair (L, v) where L is a simplicial complex and $v : L \to L$ is a simplicial involution.*

Definition 27 (Free simplicial $\mathbb{Z}_2$-complex)**.** *A simplicial complex (L, v) is called free if there is no simplex A of L such that $v(A) = A$.*

Definition 28 (Simplicial $\mathbb{Z}_2$-map)**.** *For two simplicial $\mathbb{Z}_2$-complexes (L_1, v_1) and (L_2, v_2), the map $f : L_1 \to L_2$ is called a simplicial $\mathbb{Z}_2$-map if f is a simplicial map and $f \circ v_1 = v_2 \circ f$.*

The existence of a simplicial $\mathbb{Z}_2$-map $f : L_1 \to L_2$ implies the existence of a continuous $\mathbb{Z}_2$-map $\|f\| : \|L_1\| \xrightarrow{\mathbb{Z}_2} \|L_2\|$ that is called the geometric realization of f.

Definition 29 (Common neighbors $\mathrm{CN}(A)$)**.** *For a graph $G = (V(G), E(G))$ and a subset $A \subseteq V(G)$, define common neighbors of A as*

$$\mathrm{CN}(A) = \{v \in V(G) : av \in E(G) \textit{ for all } a \in A\} \subseteq V(G) \backslash A.$$

Definition 30 (Box complex $B(G)$)**.** *Box complex $B(G)$, is a free simplicial $\mathbb{Z}_2$-complex with vertex set $V(G) \uplus V(G) = V(G) \times [2]$ and the following set of simplices*

$$\{A \uplus B : A, B \subseteq V(G), A \cap B = \varnothing, G[A, B] \textit{ is complete bipartite and } \mathrm{CN}(A) \neq \varnothing \neq \mathrm{CN}(B)\}$$

The involution is given by interchanging the two copies of $V(G)$.

Definition 31 (Box complex $B_0(G)$)**.** *Box complex $B_0(G)$, is a free simplicial $\mathbb{Z}_2$-complex with vertex set $V(G) \uplus V(G) = V(G) \times [2]$ and the following set of simplices*

$$\{A \uplus B : A, B \subseteq V(G), A \cap B = \varnothing, G[A, B] \textit{ is complete bipartite } \}.$$

The involution is given by interchanging the two copies of $V(G)$.

For more about box complex, the reader can refer to [5].

3 Results of $\chi(KG(n,k))$

Theorem 1 (Borsuk-Ulam theorem, [3])**.** *If S_k is the union of $k+1$ sets which are open in S_k, then one of these sets contains antipodal points.*

Theorem 2 (Gale's lemma, [6])**.** *If n and k are nonnegative integers, then there is a set $V \subset S_k$ with $2n+k$ elements such that $|H(a) \cap V| \geq n$ for each $a \in S_k$.*

Theorem 3 ($\chi(KG(n,k))$, [9, 2, 7])**.** *If the n-tuples of a set of $2n+k$ elements are partitioned into $k+1$ classes, then one of the classes contains two disjoint n-tuples.*

4 Results of $\chi_m(\mathrm{KG}(n,k)_s)$

Lemma 1 (Lemma 1 in [1])**.** *Let n be a positive integer, V be an n-set, and $\sigma : [n] \to V$ be a bijection. Also, let $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property and set $d = n - \mathrm{alt}(\mathcal{P}, \sigma) - 1$. If $d \neq -1$, then there are a multiset $Z \subset S^d$ of size n and a suitable identification of Z with V such that for any $x \in S^d, Z_x \in \mathcal{P}$. In particular, for $d \geq 1, Z$ can be a set.*

Revisit the proof of Lemma 1 in [1] as follows.

Proof. For simplicity of notation, assume that $V = \{v_1, \ldots, v_n\}$ where $\sigma(i) = v_i$. Consider the following curve

$$\gamma = \left\{ \left(1, t, t^2, \ldots, t^d\right) \in \mathbb{R}^{d+1} : t \in \mathbb{R} \right\}$$

and set $W = \{w_1, w_2, \ldots, w_n\}$, where $w_i = \gamma(i)$ for $i = 1, 2, \ldots, n$. Now, let $Z = \{z_1, z_2, \ldots, z_n\} \subseteq S^d$ be a set such that $z_i = (-1)^i \frac{w_i}{\|w_i\|}$ for any $1 \leq i \leq n$. Note that if $d \geq 1$, then Z is a set. Consider the identification between V and Z such that $v_i \in V$ is identified with z_i for any $1 \leq i \leq n$. It can be checked that every hyperplane of $\mathbb{R}^{d+1}$ passing trough the origin intersects γ in no more than d points. Moreover, if a hyperplane intersects the curve in exactly d points, then the hyperplane cannot be tangent to the curve; and consequently, at each intersection point, the curve passes from one side of the hyperplane to the other side.

In what follows, for any $y \in S^d$, we will show that $Z_y \in \mathcal{P}$ completing the proof. On the contrary, suppose that there is a $y \in S^d$ such that $Z_y \notin \mathcal{P}$. Let h be the hyperplane passing trough the origin that contains the boundary of $H(y)$. We can move this hyperplane continuously to a position such that it still contains the origin and has exactly d points of $W = \{w_1, w_2, \ldots, w_n\}$ while during this movement no points of W crosses from one side of h to the other side. Consequently, during the aforementioned movement, no points of $Z = \{z_1, z_2, \ldots, z_n\}$ crosses from one side of h to the other side. Hence, at each of these intersections, γ passes from one side of h to the other side. Let h^+ and h^- be two open half-spaces determined by the hyperplane h. Now consider $X = (x_1, x_2, \ldots, x_n) \in \{+, -, 0\}^n \backslash \{0\}$ such that

$$x_i = \begin{cases} 0 & \text{if } w_i \text{ is on } h \\ + & \text{if } w_i \text{ is in } h^+ \text{ and } i \text{ is even} \\ + & \text{if } w_i \text{ is in } h^- \text{ and } i \text{ is odd} \\ - & \text{otherwise.} \end{cases}$$

Assume that $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-d}}$ are nonzero entries of X, where $i_1 < i_2 < \cdots < i_{n-d}$. It is easy to check that any two consecutive terms of x_{i_j} 's have different signs. Since X has $n - d = \mathrm{alt}(\mathcal{P}, \sigma) + 1$ nonzero entries, we have $\mathrm{alt}(X) = \mathrm{alt}(-X) = \mathrm{alt}(\mathcal{P}, \sigma) + 1$; and therefore, both X_σ and $(-X)_\sigma$ are in $\mathcal{P}$. Also, one can see that either $X_\sigma \subseteq Z_y$ or $(-X)_\sigma \subseteq Z_y$. Therefore, since $\mathcal{P}$ is a signed-increasing property, we have $Z_y \in \mathcal{P}$ that is a contradiction. □

Theorem 4. *For positive integers n, k, and s with $n \geq sk$, if s is an even integer and $k \geq m$, then $\chi_m\left(\mathrm{KG}(n,k)_s\right) = n - sk + sm$.*

I think Lemma 1 in [1] could be generalized.

First, give some definitions.

Definition 32 (Length of the longest f-alternating subsequence)**.** *For an $X = (x_1, \ldots, x_n) \in \{+, -, 0\}^n$, an f-alternating subsequence of X is a subsequence of nonzero terms of X such that each of its f consecutive* *members have the same signs while the next f consecutive members have different signs compared with the previous one. In other words, $x_{j_1}, \ldots, x_{j_m}$ $(1 \le j_1 < \cdots < j_m \le n)$ is an f-alternating subsequence of X if $x_{j_i} \neq 0$ for each $i \in [m]$ and $x_{j_1} = x_{j_2} = \cdots = x_{j_f} \neq x_{j_{f+1}} = x_{j_{f+2}} = \cdots = x_{j_{2f}} \neq x_{j_{2f+1}} \cdots$ for $i = 1, \ldots, m-1$. The length of the longest f-alternating subsequence of X is denoted by $\mathrm{alt}_f(X)$. Note that $\mathrm{alt}_1(X) = \mathrm{alt}(X)$.*

Definition 33 ($\mathrm{alt}_f(\mathcal{P}, \sigma)$)**.** *Let $\sigma : [n] \to V$ be a bijection and $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property. Define*

$$\mathrm{alt}_f(\mathcal{P}, \sigma) = \max\left\{\mathrm{alt}_f(X) : X \in \{+,-,0\}^n \text{ with } X_\sigma \notin \mathcal{P}\right\}$$

Then we have the next Lemma.

Lemma 2 (A Generalization of Lemma 1 in [1])**.** *Let n be a positive integer, V be an n-set, and $\sigma : [n] \to V$ be a bijection. Also, let $\mathcal{P} \subseteq P_s(V)$ be a signed-increasing property and set $d = \lfloor \frac{1}{f}(n - \mathrm{alt}_f(\mathcal{P}, \sigma) - 1) \rfloor$. If $d \neq -1$, then there are a multiset $Z \subset S^d$ of size n and a suitable identification of Z with V such that for any $x \in S^d, Z_x \in \mathcal{P}$. In particular, for $d \ge 1, Z$ can be a set.*

Proof. For simplicity of notation, assume that $V = \{v_1, \ldots, v_n\}$ where $\sigma(i) = v_i$. Consider the following curve

$$\gamma = \left\{\left(1, t, t^2, \ldots, t^d\right) \in \mathbb{R}^{d+1} : t \in \mathbb{R}\right\}$$

and set $W = \{w_1, w_2, \ldots, w_n\}$, where $w_i = \gamma(i)$ for $i = 1, 2, \ldots, n$. Now, let $Z = \{z_1, z_2, \ldots, z_n\} \subseteq S^d$ be a set such that $z_i = (-1)^{\lfloor \frac{i}{f} \rfloor} \frac{w_i}{\|w_i\|}$ for any $1 \le i \le n$. Note that if $d \ge 1$, then Z is a set. Consider the identification between V and Z such that $v_i \in V$ is identified with z_i for any $1 \le i \le n$. It can be checked that every hyperplane of $\mathbb{R}^{d+1}$ passing trough the origin intersects γ in no more than d points. Moreover, if a hyperplane intersects the curve in exactly d points, then the hyperplane cannot be tangent to the curve; and consequently, at each intersection point, the curve passes from one side of the hyperplane to the other side.

In what follows, for any $y \in S^d$, we will show that $Z_y \in \mathcal{P}$ completing the proof. On the contrary, suppose that there is a $y \in S^d$ such that $Z_y \notin \mathcal{P}$. Let h be the hyperplane passing trough the origin that contains the boundary of $H(y)$. We can move this hyperplane continuously to a position such that it still contains the origin and has exactly d points of $W = \{w_1, w_2, \ldots, w_n\}$ while during this movement no points of W crosses from one side of h to the other side. Consequently, during the aforementioned movement, no points of $Z = \{z_1, z_2, \ldots, z_n\}$ crosses from one side of h to the other side. Hence, at each of these intersections, γ passes from one side of h to the other side. Let h^+ and h^- be two open half-spaces determined by the hyperplane h.

Now consider $X = (x_1, x_2, \ldots, x_n) \in \{+,-,0\}^n \backslash \{0\}$, assume X are first partitioned into $\lfloor \frac{n}{f} \rfloor$ parts $\{X_i\}_{i \in [\lfloor \frac{n}{f} \rfloor]}$, such that each part has f elements. Namely, $X_1 = \{x_1, \ldots, x_f\}$, $X_2 = \{x_{f+1}, \ldots, x_{2f}\}$, …, $X_{\lfloor \frac{n}{f} \rfloor} = \{x_{f\lfloor \frac{n}{f} \rfloor - f + 1}, \ldots, x_{\lfloor \frac{n}{f} \rfloor f}\}$. Correspondingly, write $W_1 = \{w_1, \ldots, w_f\}$, $W_2 = \{w_{f+1}, \ldots, w_{2f}\}$, …, $W_{\lfloor \frac{n}{f} \rfloor} = \{w_{f\lfloor \frac{n}{f} \rfloor - f + 1}, \ldots, w_{\lfloor \frac{n}{f} \rfloor f}\}$.

Let cnt_i^W record the number of elements $\{w_1, w_2, \ldots, w_{i-1}\}$ on h. Let cnt_i^X record the number of zero elements $\{x_1, x_2, \ldots, x_{i-1}\}$. Let $\mathrm{diff}_i^{(X,W)} = \lfloor \frac{\mathrm{cnt}_i^X}{f} \rfloor - \mathrm{cnt}_i^W$. Note that in the followings, $\frac{\mathrm{cnt}_i^X}{f}$ should be a integer.

Now, we assign the sign for each X_i in the following:

$$\text{all elements of } X_i = \begin{cases} + & \text{if } \mathrm{diff}_i^{(X,W)} = 0 \text{ and all } W_{\lfloor \frac{i-1}{f} \rfloor} \text{ are in } h^+ \text{ and } \mathrm{cnt}_i^W + \lfloor \frac{i-1}{f} \rfloor \text{ is even} \\ - & \text{if } \mathrm{diff}_i^{(X,W)} = 0 \text{ and all } W_{\lfloor \frac{i-1}{f} \rfloor} \text{ are in } h^+ \text{ and } \mathrm{cnt}_i^W + \lfloor \frac{i-1}{f} \rfloor \text{ is odd} \\ - & \text{if } \mathrm{diff}_i^{(X,W)} = 0 \text{ and all } W_{\lfloor \frac{i-1}{f} \rfloor} \text{ are in } h^- \text{ and } \mathrm{cnt}_i^W + \lfloor \frac{i-1}{f} \rfloor \text{ is even} \\ + & \text{if } \mathrm{diff}_i^{(X,W)} = 0 \text{ and all } W_{\lfloor \frac{i-1}{f} \rfloor} \text{ are in } h^- \text{ and } \mathrm{cnt}_i^W + \lfloor \frac{i-1}{f} \rfloor \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-fd}}$ are nonzero entries of X, where $i_1 < i_2 < \cdots < i_{n-fd}$. It is easy to check that any two f-gap consecutive terms of x_{i_j} 's have different signs, and with the f-gap are of the same sign. Since X has $n - fd \geq \mathrm{alt}_f(\mathcal{P}, \sigma) + 1$ nonzero entries, we have $\mathrm{alt}_f(X) = \mathrm{alt}_f(-X) \geq \mathrm{alt}_f(\mathcal{P}, \sigma) + 1$; and therefore, both X_σ and $(-X)_\sigma$ are in $\mathcal{P}$. Also, one can see that either $X_\sigma \subseteq Z_y$ or $(-X)_\sigma \subseteq Z_y$. Therefore, since $\mathcal{P}$ is a signed-increasing property, we have $Z_y \in \mathcal{P}$ that is a contradiction. □

When $\mathrm{alt}_f(\mathcal{P}, \sigma) \leq \mathrm{alt}(\mathcal{P}, \sigma)f + f + n - 1 - nf$, we have a larger bound for d, as $\frac{1}{f}(n - \mathrm{alt}_f(\mathcal{P}, \sigma) - 1) \geq n - \mathrm{alt}(\mathcal{P}, \sigma) - 1$.

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