# A STUDY OF DAMAGE NUMBER IN COP ROBBER GAME


## ABSTRACT

Given a graph $G$, cops and robbers can play chase on $G$. In this game, the cops pursuit the robbers with a certain goal, such as capturing the robber, in which we can define capture time (the minimum number of steps the cops need to catch the robbers). Besides, the goal of the robbers can also be to limit the range of movement of the robbers, in which we can define damage number (the maximum number of vertices the robbers can visit). In 2019, Cox and Sanaei made two conjectures: 1) For any given $r \in (0, 1)$, there is a series of graphs such that the ratio of the damage number and the capture time can approach $r$. 2) For Paley graph $\mathcal{P}_n$, the damage number equals $\frac{n-1}{2}$ when $n > 9$. We proved Conjecture 1 and we also proved Conjecture 2 in the special case that $n = 13$. We introduced and discussed the concepts of capture-and-smallest-damage number.


**Key words:** Cop Robber Game, Damage Number, Capture Time

# Contents

# Chapter 1 Introduction

In the real world, humans will inevitably encounter scenarios of pursuit and evasion. For example, tracking by Sensor Actuator Network (SAN) can be modeled as a pursuit-evasion game, where the object being tracked is trying to evade by moving in a way that prevents the SAN from observing it. Other scenarios include surveillance, search-and-rescue, and quality inspection. There are two advantages to formulate these tasks as a pursuit-evasion game. The game-theoretical formulation naturally captures the adversarial nature of these tasks[1]. Thus, it is meaningful to study the pursuit-evasion game. The pursuit-evasion game is a vertex pursuit game, where the environment is represented by a graph.

The *Cop and Robber* game is a specific vertex pursuit game. In this game, a simple, undirected and reflexive graph $G = (V, E)$ is given, player I, known as the cop, wants to cat player II, known as the robber. First, the cop chooses a vertex to start, then the robber chooses a vertex. At each round, the cop moves first. The cop can move from one vertex to the adjacent vertex or pass the round, i.e., stay at the original vertex. Then, the robber moves from one vertex to the adjacent vertex or passes the round. In each round, both the two players are aware of the other's positions and they can design their moving strategies. The rule of the game is that, if the robber is caught after a finite number of moves, the cop wins the game, otherwise, the robber wins. As the *Cop and Robber* is a game of complete information, there is a winning strategy for the cop or the robber. If $G$ is a graph the cop has a winning strategy, $G$ is called a *cop-win* graph, otherwise, $G$ is called a *robber-win* graph.

There are other settings of the *Cop and Robber* game. For example, the game with $k$ cops and $l$ robber, where $k > 1$, where the minimum number of cops who guarantees the cop's winning strategy on a graph $G$ (the cop number of $G$) is studied[2]. And the games played on directed graphs[1] or game with partial information[3, 4]. For more versions, see the surveys[5, 6].

Recently, other variations of the *Cop and Robber* game have been studied. For example,[4] introduce a new graph parameter called *burning number*, which measures the speed of the spread of contagion in a graph. Besides, Clarke et al.[7] propose a new model on directed acyclic graphs, in which the contamination spreads slowly. Cox and Sanaei[8] introduces the *damage number* of a graph, which is the minimum number of distinct vertices the robbers can visit without capture. The motivation of the proposed *damage number* is that in some situations, the damage by the intruder is severe or costly and the highest priority is to contain the damage instead of capturing the intruder.

In this paper, we will first introduce some remarkable results in *cop-win* graph and *damage number*. Then, we present our proof and analysis for the previous problems about *damage number*. Furthermore, we propose a new meaningful parameter *capture-and-smallest-damage number*, which stands for the minimal capture time in *cop-win* graph with the smallest damage number.

# Chapter 2 Definition

We give definitions appearing in this paper to refer as follows.

**Definition 2.1** $N(v)$: $\{v' \in V | (v, v') \in E\}$, the neighbourhood of $v$.

**Definition 2.2** $N[v]$: $N(v) \bigcup \{v\}$, the closed neighbourhood of $v$.

**Definition 2.3** $\mathtt{deg}(v)$: $|N(v)|$, the degree of vertex $v$.

**Definition 2.4** $\mathtt{dis}(u, v)$: the distance of vertex $u$, $v$.

**Definition 2.5** $\Delta(G)$: $\max\{\mathtt{deg}(v) | v \in V(G)\}$, the maximum degree of vertex in graph $G$.

**Definition 2.6** $\mathtt{ecc}(v)$: $\max\{\mathtt{dis}(u, v) | u \in V(G)\}$, the eccentric distance of vertex $v$ in graph $G$.

**Definition 2.7** $\mathtt{rad}(G)$: $\min\{\mathtt{ecc}(v) | v \in V(G)\}$, the radius of graph $G$.

**Definition 2.8** Corner: a vertex $v \in V(G)$ such that $N[v] \subseteq N[u]$ for some $v \in V(G)$.

**Definition 2.9** $\mathtt{capt}(G)$: the minimum number of cop's moves to capture the robber in the graph $G$, no matter how the robber moves.

**Definition 2.10** $\mathtt{dmg}(G)$: the maximum number of vertices of $G$ that the robber can damage, no matter how the cop moves.

**Definition 2.11** Dismantlable graph $DG(k)$: a graph with $k$ separate corners $v_1, \cdots, v_k$ and $G - \{v_1, \cdots, v_k\}$ is also $DG(k)$.

**Definition 2.12** Strong regular graph $SRG(n, k, \lambda, \mu)$: a graph with $n$ vertices, each having $k$ neighborhoods. Every two connected vertices have $\lambda$ common neighbors, two disconnected vertices $\mu$ common neighbors.

# Chapter 3 Cop-win Graph

In this section, we introduce some results of *cop-win* graph. First, there are some lemmas.

**Lemma 3.1** A cop-win graph is a graph with connected component 1.

**Lemma 3.2 (Triangle Lemma)** If $G$ is a cop-win graph, then each edge $(u, v) \in E(G)$ is either a bridge or $u$ and $v$ have a common neighbor $w$, which forms a triangle over $(u, v)$.

**Proof** If $(u, v)$ is not a bridge, and $u$ and $v$ don't have a common neighbor, then there is a $C_4$ in $G$. Besides, no vertex connects 3 vertices of this $C_4$. Thus, the robber can move to stay at the diagonal vertex of the cop. □

There are some remarkable theorems of *cop-win* graph. We present them as follows with brief proof ideas.

**Theorem 3.3 (Nowakowski and Winkler[9])** A retract of a cop-win graph is cop-win.

**Proof** Assume that $G'$ is the retract of $G$. The cop just follows the strategy in $G$. □

Although the proof of this theorem seems to be simple, it leads to some important corollaries.

**Corollary 3.4** If $G$ is cop-win and $v$ is a corner in G, then $G - v$ is cop-win.

**Corollary 3.5** If $G$ is cop-win and $G'$ is also cop-win by removing a corner of $G$, then $\texttt{capt}(G) \leq \texttt{capt}(G') + 1$.

**Theorem 3.6 (Bonato et al.[10])** If $G$ is a finite $DG(2)$, $\texttt{capt}(G) \leq \lfloor \frac{n}{2} \rfloor$.

**Proof** Induct on $n$. Each time select two vertices because of 2-dismantable property. Then get a retract of the graph and use inductive assumption. □

**Theorem 3.7 (Gavenčiak[11])** If $n \geq 7$, $\texttt{capt}_{\texttt{max}}(n) = n - 4$, and for $n \leq 7$, $\texttt{capt}_{\texttt{max}}(n) = \lfloor \frac{n}{2} \rfloor$.

**Proof** (1) For $n \leq 7$, enumerate all possible $n$, we have:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathtt{capt}_{\mathtt{max}}(n)$ | 0 | 1 | 1 | 2 | 2 | 3 | 3 |

.

(2) For $n \geq 8$, the lower bound is guaranteed by using the corollary 3.5 to induct on $n$. The upper bound is guaranteed by constructing a family of graphs $H_n$ in Figure 3–1 which has $\mathtt{capt}_{\mathtt{max}}(H_n) = n - 4$. □

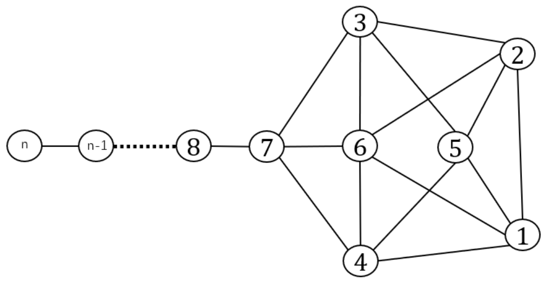


Figure 3–1 Graph $H_n$.

Note that this theorem improves the results in the previous paper[10] Theorem 2.

# Chapter 4 Damage Number

The following results about *damage number* are all from the previous work[8]. We indicate briefly ideas of their proofs.

**Theorem 4.1** If $n \geq 4$, $\mathtt{dmg}(C_n) = \lfloor \frac{n-1}{2} \rfloor$.

**Proof** (1) Cop can protect $\lceil \frac{n-1}{2} \rceil$: when the robber passes, passes as the same; when the robber moves, moves at the opposite direction. (2) Robber can damage $\lfloor \frac{n-1}{2} \rfloor$: starts as close to cop as possible, then moves at clockwise. □

**Theorem 4.2** $\lfloor \frac{\mathtt{rad(G)}}{2} \rfloor \leq \mathtt{dmg}(G) \leq n - \Delta(G) - 1$.

**Theorem 4.3** If $n \leq 8$, then $\mathtt{dmg}(G) \leq \lfloor \frac{n}{2} \rfloor$.

**Proof** For $n \leq 8$, enumerate all possible $n$, we have:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\mathtt{dmg}(G)$ | 0 | 0 | 0 | 1 | 2 | 2 | 3 | 4 |
| Example | - | - | - | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $M_8$ |

where $M_8$ is the Mӧbius ladder $M_8$, as shown in Figure 4–1. □

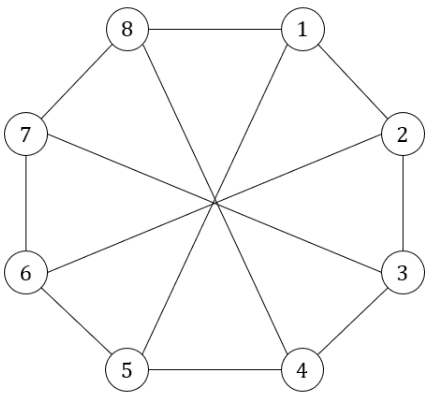


Figure 4–1 Mӧbius ladder $M_8$.

**Theorem 4.4** For the Mӧbius ladder $M_8$, $\mathtt{dmg}(M_8) = 4$.

**Proof** (1) It is clear that the cop can protect 4 vertices by passing all the rounds. (2) The robber can damage three vertices in the first three consecutive moves. It can be proved by enumerating the first three steps. (3) The robber can damage the fourth vertex by discussing case by case. □

Some graph has *damage number* greater than half of $|V|$.

**Theorem 4.5** For the Pappus graph $G$ shown in Figure 4–2, $\mathrm{dmg}(G) \geq 10$.

**Proof** (1) If the robber is not forced by the cop to complete a cycle, then the robber can damage at least ten vertices. This is because of three facts, the Pappus graph is bipartite, the robber can access three vertices and the cop can only protect one vertex in a round. (2) If the robber is forced by the cop to complete a cycle, we only need to consider 6-cycle and 8-cycle. We can finish the proof by enumerating all possible cases. □

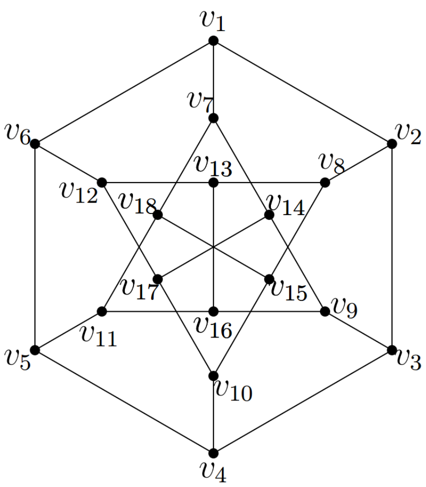

Figure 4–2 The Pappus graph.

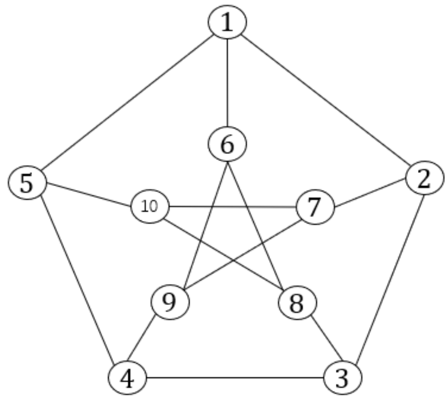

Figure 4–3 The Peterson graph.

**Theorem 4.6** For the Peterson graph $G$ shown in Figure 4–3, $\mathrm{dmg}(G) = 5$.

**Proof** (1) The robber can damage five vertices in the first five consecutive moves. (2) The cop can protect five vertices by force the robber to damage a 5-cycle and then protect the others. □

**Theorem 4.7** For the graph $G = SRG(n, k, \lambda, \mu)$, $\mathrm{dmg}(G) \geq \min\{k - \lambda, k - \mu + 1\}$.

**Proof** Thinking backward, assume that the robber is already not able to damage any vertex. □

**Theorem 4.8** For a Paley graph $\mathcal{P}_n$, $n > 9$, $\frac{n+3}{4} + 1 \leq \mathrm{dmg}(\mathcal{P}_n) \leq \frac{n-1}{2}$.

**Proof** Utilize that the maximum clique in a Paley graph of order $n$ has $\sqrt{n}$ vertices. □

# Chapter 5 Main Results

## 5.1 Damage Number of Paley Graph

**Problem description:** *For a Paley graph $\mathcal{P}_n$, decide whether* $\mathtt{dmg}(\mathcal{P}_n) = \frac{n-1}{2}$ *for* $n > 9$ *or not.*

In graph theory, Paley graphs are a family of dense undirected graphs. The number of vertices $|V|$ is a prime power and $|V| \equiv 1 \pmod 4$. Let vertex set $V = \{v_0, v_1, \cdots, v_{|V|-1}\}$, the edge set $E = \{(v_i, v_j) | i - j \in (\mathbf{F}_{|V|}^{\times})^2\}$. This means that two vertices connect if and only if the difference of their indices is a quadratic residue of $|V|$.

Note that we already have the the results for $n = 2$ and $n = 3$, i.e., $\mathtt{dmg}(\mathcal{P}_5) = 2$ and $\mathtt{dmg}(\mathcal{P}_9) = 3$.

**Theorem 5.1** For the Paley graph $\mathcal{P}_9$, $\mathtt{dmg}(\mathcal{P}_9) = 3$.

**Proof** Consider $\mathcal{P}_n = C_3 \square C_3$ in Figure 5–1, where $\square$ is the Cartesian product of graphs. □

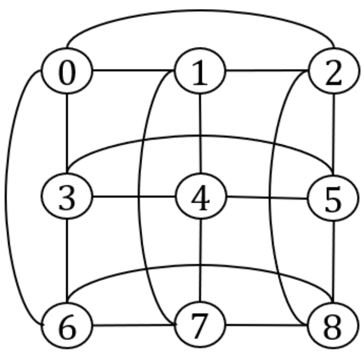


Figure 5–1 Graph $\mathcal{P}_9$ as $C_3 \square C_3$.

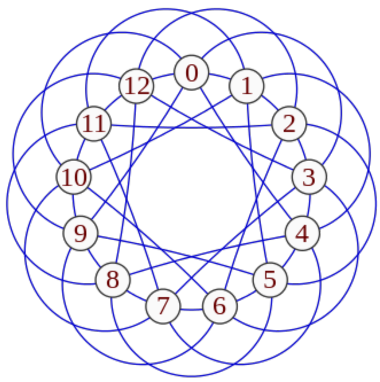


Figure 5–2 Graph $\mathcal{P}_{13}$. The image is from Wikipedia.

**Theorem 5.2** For the Paley graph $\mathcal{P}_{13}$, $\mathtt{dmg}(\mathcal{P}_{13}) = 6$.

**Proof** First, notice that it is already proved that the damage number of $\mathcal{P}_{13} \geq 5$ in Theorem 4.8. Thus, we only need to prove that the robber can damage another more vertex. To prove it, we notice the graph $\mathcal{P}_{13}$ in Figure 5–2. We find that for any vertex

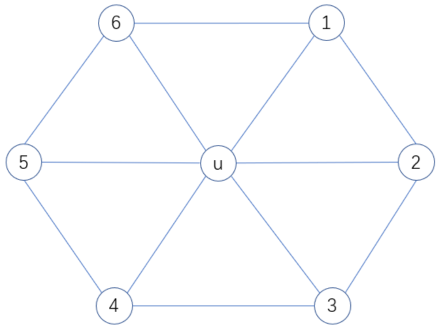


Figure 5–3 Graph $A_6$.

$u$ in graph $\mathcal{P}_{13}$, it has its 6 neighbors like $A_6$ in Figure 5–3. This result is crucial to complete the proof.

Assume that the robber has already damaged 4 vertices $u_i, (i = 1, 2, 3, 4)$, and he is now on the vertex $u_5$. It is the robber's move. As the robber has 6 neighbors, he must has two neighbors different from $u_i, (i = 1, 2, 3, 4)$, we name these two neighbors as $v_1, v_2$. There are two big cases to consider: 1) the cop is not adjacent to the robber; 2) the cop is adjacent to the robber. We discuss these two cases in the following.

*Case 1:* the cop is not adjacent to the robber. First, it is impossible for $u_5$ adjacent to all the $u_i, (i = 1, 2, 3)$. If $u_5$ is adjacent to $u_i, (i = 1, 2, 3, 4)$, as $w$ and $u_5$ has 3 common neighbors, we assume that the third neighbor is $u_1$. To form the $A_6$ of $u_5$, we assume that $(v_1, u_4), (v_1, v_2), (v_2, u_1)$ are connected. This is shown in Figure 5–4. However, in this case, $v_2$ can not form $A_6$. Thus, it is impossible for $w$ to connect $v_1, v_2$.

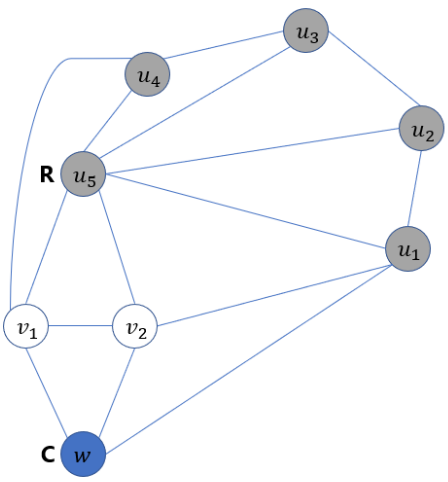


Figure 5–4 Case 1.

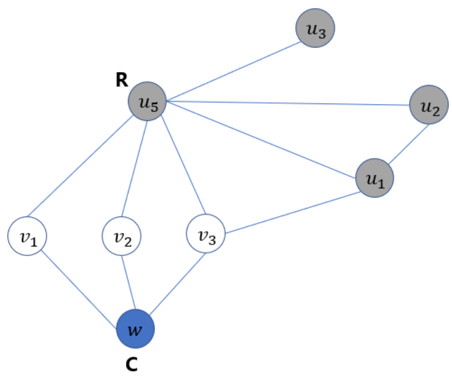


Figure 5–5 Case 1.

Thus, $u_5$ is not adjacent to all the $u_i, (i = 1, 2, 3)$. Then $u_5$ connects to another undamaged vertex $v_3$. We can assume that $u_5$ is connect to three vertices of $u_i, (i = 1, 2, 3)$. Otherwise, $u_5$ is connected to at least 4 vertices, but $w$ can only protect 3 vertices. We assume that $u_5$ connects to $u_1, u_2, u_3$. As $u_1$ should connect one of $v_i, (i = 1, 2, 3)$, we assume $u_1$ connects to $v_3$. And $v_i, (i = 1, 2, 3)$ should have and only have an edge. Otherwise, if more than one edge, one vertex of $v_i, (i = 1, 2, 3)$ can not form $A_6$. We get the graph shown in Figure 5–5. As $v_1, v_2$ is equivalent, we have two cases to consider.

*Case 1.1:* $v_2$ connects to $v_3$. In this case, $v_2$ is not connected to $v_1$. Then $v_2$ connects to $u_3$ and $(v_1, u_3)$, $(v_1, u_2)$ is connected. If one of $u_1, u_2, u_3$ can have 3 new neighbors different from $u_4$ (the damaged vertex). Then the robber can move to that vertex $x \in \{u_1, u_2, u_3\}$. No vertex can protect these three undamaged neighbors of $x$, otherwise one of the neighbor of $x$ can not form $A_6$. Figure 5–6 is an example. In Figure 5–6, $x = u_1$ and $x$ has three new undamaged neighbors $k_1, k_2, k_3$. As $u_1$ forms the $A_6$, if one vertex connects to all the $k_1, k_2, k_3$ to protect them, them $k_2$ can not form $A_6$.

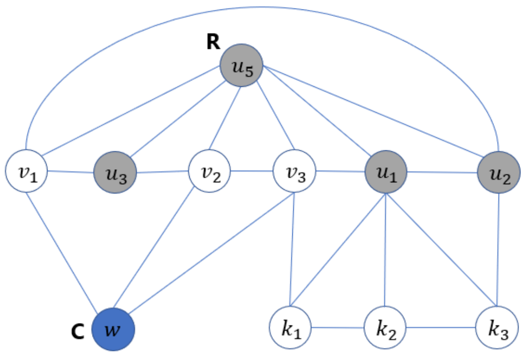


Figure 5–6 Case 1.1.

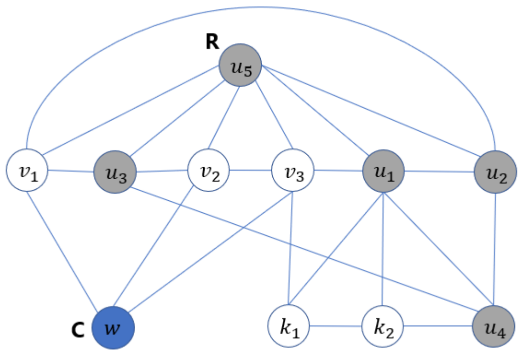


Figure 5–7 Case 1.1.

If $u_1, u_2, u_3$ all connect to $u_4$, we consider $u_1$. $u_1$ can connect to two undamaged vertices $k_1, k_2$ and form $A_6$. This situation is like the above, no vertex can protect both $v_3, k_1, k_2$, otherwise $k_1$ can not form $A_6$. This is shown is Figure 5–7.

*Case 1.2:* $v_2$ connects to $v_1$. This is similar to *Case 1.1*, as shown in Figure 5–8 and Figure 5–9.

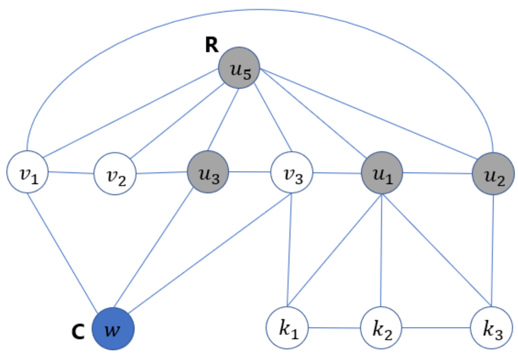


Figure 5–8 Case 1.2.

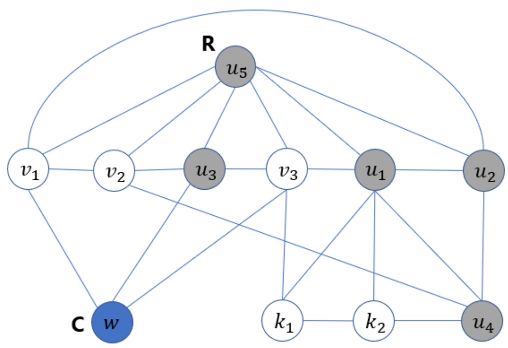


Figure 5–9 Case 1.2.

*Case 2:* the cop is adjacent to the robber. Considering whether $w$ is damaged, we have two cases.

*Case 2.1:* $w$ is not damaged. Assume that $u_5$ connects to the damaged vertices $u_1, u_2, u_3$. If one of $u_1, u_2, u_3$ can have 3 new neighbors different from $u_4$ (the damaged vertex). Then we can prove as the above. Assume that $u_1, u_2, u_3$ all connect to $u_4$. For $u_1$, it has two undamaged neighbors $k_1, k_2$. When $u_1$ forms $A_6$, no vertex can protect both $v_2, k_1, k_2$, otherwise $k_1$ can not form $A_6$. This is shown in Figure 5–10.

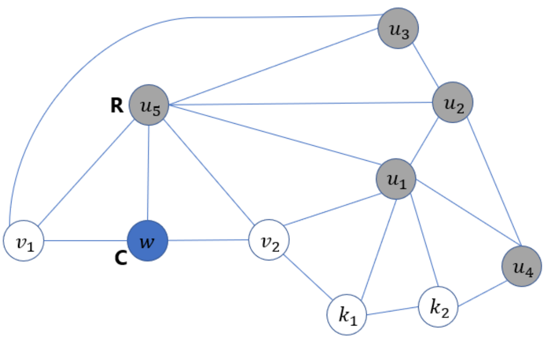


Figure 5–10 Case 2.1.

*Case 2.2:* $w$ is damaged. Assume that $u_5$ connects to the damaged vertices $u_1, u_2, u_3$. And we can prove like *Case 2.1*.

The core of the above proof is the structure of $A_6$ in $\mathcal{P}_{13}$. Therefore, we also present a similar structure in $\mathcal{P}_{17}$ in Figure 5–11, which we define as $A_8$. However, we find that $A_8$ seems to be not enough to decide whether $\mathtt{dmg}(\mathcal{P}_{17}) = 8$ or not.

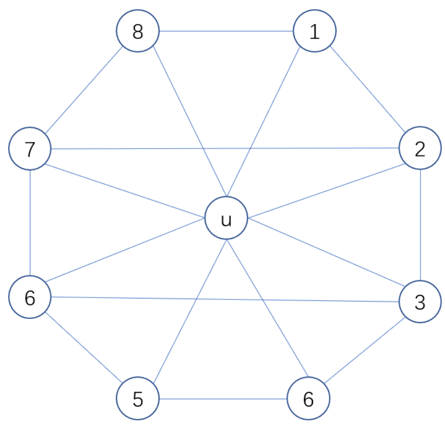


Figure 5–11 Graph $A_8$.

## 5.2 The Ratio of Damage Number and Capture Time

**Problem description:** $\forall r \in (0, 1)$, *decide whether there is a series of graph $G_n$, such that* $\frac{\mathtt{dmg}(G_n)}{\mathtt{capt}(G_n)} \longrightarrow r \quad (n \longrightarrow \infty)$.

From the previous work, we can easily get the results of $r = \frac{1}{2}$ and $r = 1$.

**Theorem 5.3** When $r = \frac{1}{2}$, for graph $H_n$, $\frac{\mathtt{dmg}(H_n)}{\mathtt{capt}(H_n)} \longrightarrow \frac{1}{2} \quad (n \longrightarrow \infty)$.

**Proof** From the previous paper[8] Theorem 4.1, we have $\mathtt{dmg}(H_n) = \lfloor \frac{n-1}{2} \rfloor - 2$. From[11] Lemma 7, we have $\mathtt{capt}(H_n) = n - 4$. Combining these two results, we get $\frac{\mathtt{dmg}(H_n)}{\mathtt{capt}(H_n)} = \frac{\lfloor \frac{n-1}{2} \rfloor - 2}{n-4} \longrightarrow \frac{1}{2} \quad (n \longrightarrow \infty)$. □

**Theorem 5.4** When $r = 1$, for graph $P_n$, $\frac{\mathtt{dmg}(P_n)}{\mathtt{capt}(P_n)} \longrightarrow 1 \quad (n \longrightarrow \infty)$.

**Proof** As $\mathtt{capt}(P_n) = \lfloor \frac{n}{2} \rfloor = \mathtt{dmg}(P_n) + 1$, we have $\frac{\mathtt{dmg}(P_n)}{\mathtt{capt}(P_n)} = \frac{\lfloor \frac{n}{2} \rfloor - 1}{n-4} \longrightarrow 1 \quad (n \longrightarrow \infty)$. □

For $r \in [0, 1)$, we use the graph $H_n$ proposed in the work[11], as shown in Figure 3–1. We construct the chain of $H_n$, named as $CH(m_1, k_1, m_2, k_2)$, where $m_1, k_1, m_2, k_2$ are positive integers, *s.t.* $4m_1 + k_1 m_1 \geq 4m_2 + k_2 m_2$. A graph $CH(m_1, k_1, m_2, k_2)$ contains $m_1$ $H_7$ heading left, $m_2$ $H_7$ heading right, and $k_1 \times m_1$ nodes in the left chain and $k_2 \times m_2$ nodes in the right chain. To make it more clear, we present this graph in Figure 5–12.

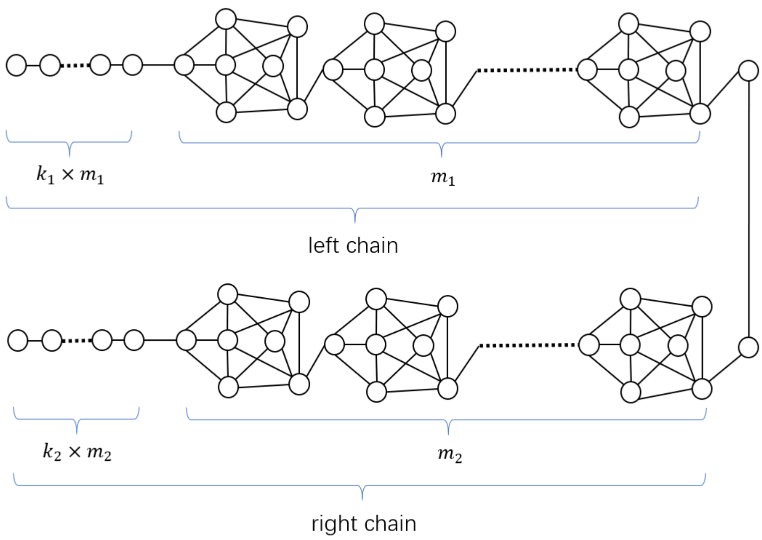


Figure 5–12 Graph chain of $H_n$, $CH(m_1, k_1, m_2, k_2)$.

**Theorem 5.5** For the graph $CH(m_1, k_1, m_2, k_2)$, where $m_1, k_1, m_2, k_2$ are positive integers, $s.t.$ $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\mathtt{dmg}(CH)}{\mathtt{capt}(CH)} = \frac{1}{2}(\frac{3+k_1}{4+k_1} + \frac{3+k_2}{4+k_1}t)$ , where $t = \frac{m2}{m1}$.

**Proof** Notice that in $H_7$ shown in Figure 3–1 where $n = 7$, if the robber is in vertex 1, and the cop in a vertex outside this $H_7$ and connected to vertex 1, which we set vertex $x$. And if $x$ is not connect to any vertex in this $H_7$ and the output vertex in this $H_7$ is vertex 7. Then, the maximal capture time used in this $H_7$ is 4. That is $x \to 1 \to 2 \to 3 \to 7$. Besides, the maximal damage number within this $H_7$ is 3. That is $1 \to 6 \to 7$.

Thus, we have:

$$\mathtt{capt}(CH) = 4m_1 + k_1m_1 \tag{5–1}$$

$$\mathtt{dmg}(CH) = \frac{1}{2}(3m_1 + k_1m_1 + 3m_2 + k_2m_2) \tag{5–2}$$

As $t = \frac{m2}{m1}$, we have:

$$\begin{aligned} \frac{\mathtt{dmg}(CH)}{\mathtt{capt}(CH)} &= \frac{1}{2}\frac{3m_1 + k_1m_1 + 3m_2 + k_2m_2}{4m_1 + k_1m_1} \\ &= \frac{1}{2}(\frac{3 + k_1}{4 + k_1} + \frac{3 + k_2}{4 + k_1}t) \end{aligned} \tag{5–3}$$

This completes the proof. □

**Theorem 5.6** For the graph $CH(m_1, k_1, m_2, k_2)$, where $m_1, k_1, m_2, k_2$ are positive integers, $s.t.$ $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\mathtt{dmg}(CH)}{\mathtt{capt}(CH)} \longrightarrow r \in (\frac{3}{8}, 1) \quad (n \longrightarrow \infty)$, where $n$ is the total number of vertices.

**Proof** Because $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have:

$$0 \leq t \leq \frac{4 + k_1}{4 + k_2} \tag{5–4}$$

Thus, by Theorem 6.3, we have:

$$\frac{\mathtt{dmg}(CH)}{\mathtt{capt}(CH)} \longrightarrow r \in (\frac{3}{8}, 1) \tag{5–5}$$

This completes the proof. □

Similarly, we construct the graph modified chain of $H_n$, named as $MCH(m_1, k_1, m_2, k_2)$, where $m_1, k_1, m_2, k_2$ are positive integers, $s.t.$ $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$. This graph is shown in Figure 5–13. Note that the vertex connected with bidirectional arrow is the same vertex.

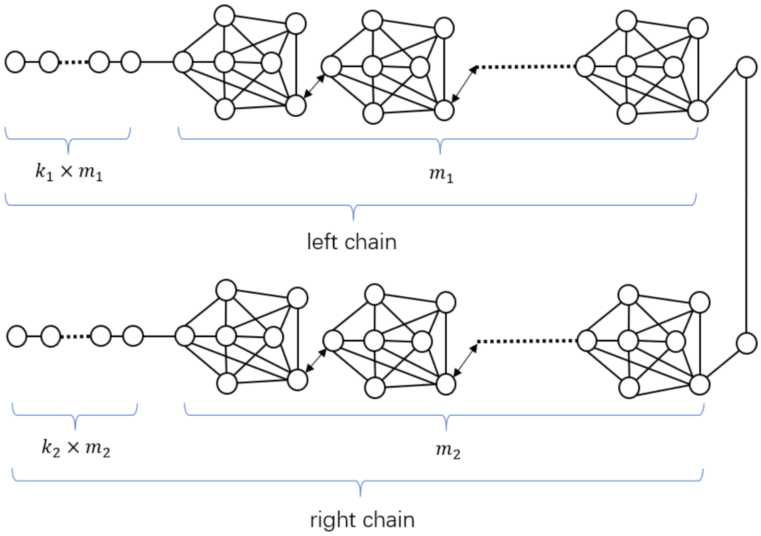


Figure 5–13 Graph modified chain of $H_n$, $MCH(m_1, k_1, m_2, k_2)$.

**Theorem 5.7** For the graph $MCH(m_1, k_1, m_2, k_2)$, where $m_1, k_1, m_2, k_2$ are positive integers, *s.t.* $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\mathtt{dmg}(MCH)}{\mathtt{capt}(MCH)} \longrightarrow r \in (\frac{1}{8}, 1)$ $(n \longrightarrow \infty)$, where $n$ is the total number of vertices.

**Proof** The proof of this theorem is similar to the proofs of Theorem 5.5 and Theorem 5.6. Notice that in each block similar to $H_7$, the robber can only damage one vertex instead of three vertices. □

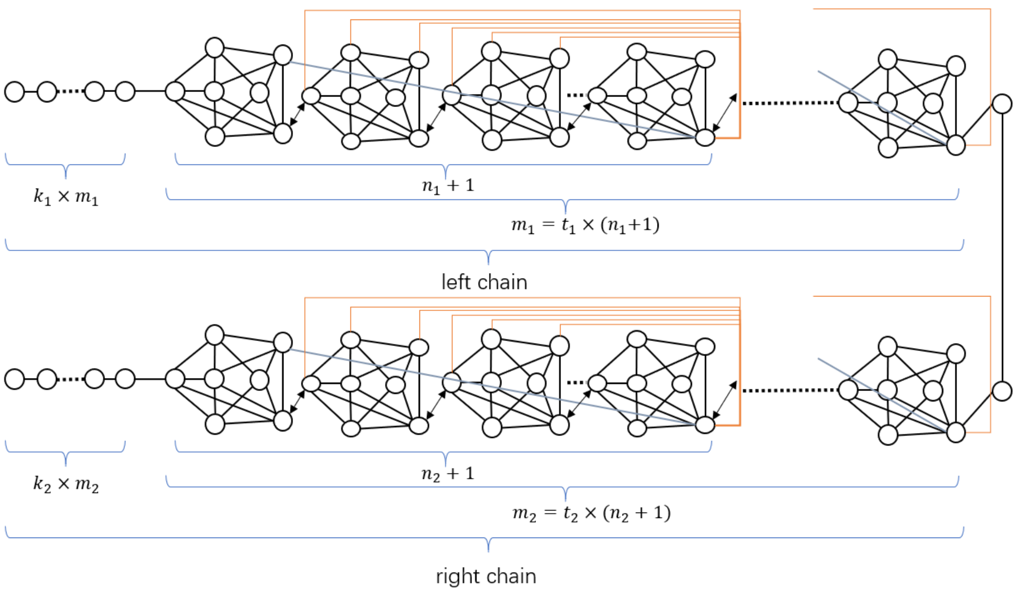


Figure 5–14 Graph modified high-way chain of $H_n$, $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$.

Finally, we use the "high-way" connection to get the graph modified high-way chain of $H_n$, i.e $MHCH$ shown in Figure 5–14.

**Theorem 5.8** For the graph modified high-way $H_n$, $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, $s.t.$ $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\mathrm{dmg}(MHCH)}{\mathrm{capt}(MHCH)} = \frac{1}{2}\frac{\frac{3}{n_1}+k_1+(\frac{3}{n_2}+k_2)t}{4+k_1}$, where $t = \frac{m2}{m1}$.

**Proof** To prove this theorem, we need to notice that if the robber wants to damage more vertices, he needs to escape by the "high-way", i.e. the pink line in Figure 5–14. Otherwise, he will be stuck in one of $n_1 + 1$ blocks and can not damage the vertices outside the block. □

**Theorem 5.9** For the graph modified high-way $H_n$, $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, $s.t.$ $4m_1 + k_1m_1 \geq 4m_2 + k_2m_2$, we have $\frac{\mathrm{dmg}(MHCH)}{\mathrm{capt}(MHCH)} \longrightarrow r \in (0, 1) \quad (n \longrightarrow \infty)$, where $n$ is the total number of vertices.

**Proof** Utilizing Theorem 5.8, it is easy to prove this theorem. □

# Chapter 6 Capture-and-smallest-damage Number

Motivated by Cox and Sanaei[8], we introduce a new parameter of the *cop-win* graph related to *damage number*, the *capture-and-smallest-damage number*. We notate the *capture-and-smallest-damage number* of graph $G$ as $\mathtt{dmg}_{\mathtt{capt}}(G)$. In some real-world scenarios, we need to achieve the goal of capturing the robber and limit the damage done by the robber to the minimum. Mathematically, we need to design a graph that is *cop-win* and the damage number is small after capturing the robber.

It is not easy. For example, in $H_n$, $\mathtt{capt}(H_n) = n-4$ and $\mathtt{dmg}(H_n) = \lfloor \frac{n-1}{2} \rfloor - 2$, but $\mathtt{dmg}_{\mathtt{capt}}(H_n)$ is not small.

**Theorem 6.1** For the graph $H_n$, $\mathtt{dmg}_{\mathtt{capt}}(H_n) = \mathtt{capt}(H_n) - 1 = n - 5$.

**Proof** Because the cop needs to capture the robber instead of limit the robber within vertex 5 in Figure 3–1, thus the cop needs to use the strategy starting at vertex 1 or 2.□

Note that $\mathtt{dmg}_{\mathtt{capt}}(H_n)$ is much greater than $\mathtt{dmg}(H_n)$, because the cop needs to capture the robber in the end. This goal limits the strategy of the cop. Generally we have:

**Theorem 6.2** For any graph $G$, $\mathtt{dmg}(G) \le \mathtt{dmg}_{\mathtt{capt}}(G) \le \mathtt{capt}(G) - 1$.

For graph $MHCH$, we have:

**Theorem 6.3** For the graph modified high-way $H_n$, $MHCH(m_1, k_1, n_1, m_2, k_2, n_2)$, where $m_1, k_1, n_1, m_2, k_2, n_2$ are positive integers, *s.t.* $4m_1 + k_1 m_1 \ge 4m_2 + k_2 m_2$, we have $\mathtt{dmg}_{\mathtt{capt}}(MHCH) = \mathtt{capt}(MHCH) - 1$.

**Proof** This theorem can be easily proved because the cop needs to capture the robber in the end. □

We believe that it is meaningful to find or construct some graph $G$ that $\mathtt{dmg}_{\mathtt{capt}}(G)$ is small. We leave it as future work.

# Chapter 7 Conclusion and Open Problems

In this undergraduate thesis, we study some problems of damage number in cop and robber game. The following are two new results we obtained.

**Result 1:** We constructed a series of graphs that can satisfy an abstract ratio of damage number dividing capture time.

**Result 2:** We proved that Paley graph $\mathcal{P}_{13}$ has damage number 6.

Finally, there are some open problems to explore in the following.

**Problem 1:** Determine whether the damage number of Paley graph $\mathcal{P}_n$ is extreme when $n \geq 17$.

**Problem 2:** Determine whether there are a series of graphs, such that their capture-and-smallest-damage number is much smaller than the capture time.

**Problem 3:** Determine the capture-and-smallest-damage number for $k$ cops and 1 robber game, where $k > 1$.

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